Unobserved Confounding, Bounds, and Sensitivity Analysis

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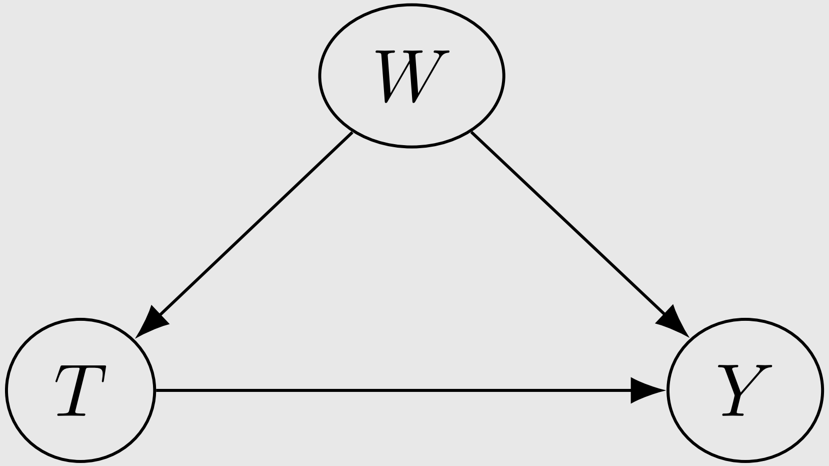
causalcourse.com

Motivation: Unobserved Confounding



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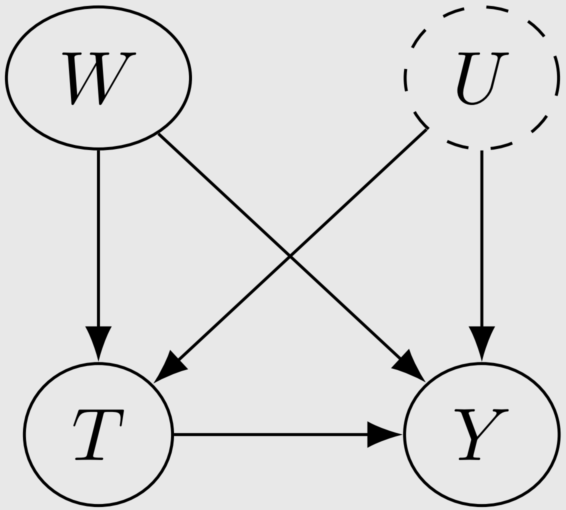

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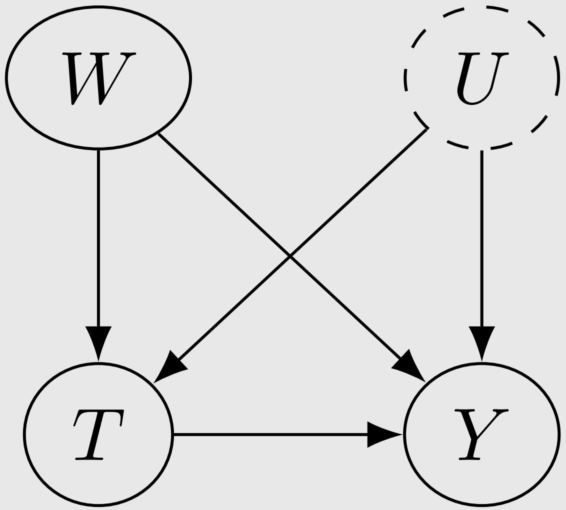

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$$\stackrel{?}{\approx} \mathbb{E}_{W}\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right]$$

Bounds

No-Assumptions Bound

Monotone Treatment Response

Monotone Treatment Selection

Optimal Treatment Selection

Sensitivity Analysis

Linear Single Confounder

Towards More General Settings

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“Partial identification” or “set identification”

Bounded Potential Outcomes

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Observational-Counterfactual Decomposition

Proof:

$\mathbb{E}[Y(1) - Y(0)]$

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\mathbb{E}[Y(1) - Y(0)] &= \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)] \\
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Observational

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&= P(T=1)\,\underline{\mathbb{E}[Y \mid T=1]} + P(T=0)\,\mathbb{E}[Y(1) \mid T=0] \\
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Observational-Counterfactual Decomposition

$$\mathbb{E}[Y(1) - Y(0)] = \pi\, \mathbb{E}[Y \mid T = 1] + (1 - \pi)\, \mathbb{E}[Y(1) \mid T = 0] - \pi\, \mathbb{E}[Y(0) \mid T = 1] - (1 - \pi)\, \mathbb{E}[Y \mid T = 0]$$

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$$\mathbb{E}[Y(1) - Y(0)] \leq \pi\,\mathbb{E}[Y \mid T=1] + (1-\pi)\,b \qquad - (1-\pi)\,\mathbb{E}[Y \mid T=0]$$

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Recall trivial bound: $a - b \leq \underbrace{\mathbb{E}[Y(1) - Y(0)] \leq}_{} b - a$

Trivial length limit: $2(b - a)$

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No-assumptions interval length: $(1 - \pi)\,b + \pi\,b - \pi\,a - (1 - \pi)\,a = b - a$

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Trivial length limit: $2(b - a)$

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$$\mathbb{E}[Y(1) - Y(0)] \leq \pi\, \mathbb{E}[Y \mid T = 1] + (1 - \pi)\, b - \pi\, a - (1 - \pi)\, \mathbb{E}[Y \mid T = 0]$$

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No-assumptions interval length: $(1 - \pi)\, b + \pi\, b - \pi\, a - (1 - \pi)\, a = b - a$

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Plugging these in to the no-assumptions bound:

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Questions:

1. What kind of bounds can we get on the ATE if the potential outcomes are unbounded?
2. Assuming bounded potential outcomes, how much smaller of an interval can we get than the trivial interval [a – b, b – a]?
3. Re-derive the Observational-Counterfactual Decomposition.
4. Derive a more general no-assumptions bound where $a_1 \leq Y(1) \leq b_1$ and $a_0 \leq Y(0) \leq b_0$.

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Combining MTR lower bound with no-assumptions upper bound:

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$$\forall i \; Y_i(1) \leq Y_i(0)$$

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No-assumptions bound: $-0.17 \leq \mathbb{E}[Y(1) - Y(0)] \leq 0.83$

Combining nonpositive MTR upper bound with no-assumptions lower bound:

$$-0.17 \leq \mathbb{E}[Y(1) - Y(0)] \leq 0$$

Question:

Given, the nonpositive MTR assumption, prove $\mathbb{E}[Y(1) - Y(0)] \leq 0$.

Monotone Treatment Selection (MTS)

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Treatment groups' potential outcomes are better than control groups':

$$\mathbb{E}[Y(1) \mid T = 1] \geq \mathbb{E}[Y(1) \mid T = 0], \quad \mathbb{E}[Y(0) \mid T = 1] \geq \mathbb{E}[Y(0) \mid T = 0]$$

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Under the MTS assumption, the ATE is bounded from above by the associational difference. Mathematically,

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Question: Prove the above MTS upper bound.

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Combining MTS upper bound with no-assumptions lower bound:

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Adding nonnegative MTR assumption and combining MTS upper bound with MTR lower bound ($\mathbb{E}[Y(1) - Y(0)] \geq 0$):

$$0 \leq \mathbb{E}[Y(1) - Y(0)] \leq 0.7$$

Bounds

No-Assumptions Bound

Monotone Treatment Response

Monotone Treatment Selection

Optimal Treatment Selection

Sensitivity Analysis

Linear Single Confounder

Towards More General Settings

Optimal Treatment Selection (OTS) Assumption

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(Observational-Counterfactual Decomposition)

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&\quad - \pi\, \mathbb{E}[Y(0) \mid T=1] - (1-\pi)\, \mathbb{E}[Y \mid T=0] \\
&\leq \pi\, \mathbb{E}[Y \mid T=1] + (1-\pi)\, \mathbb{E}[Y \mid T=0] \\
&\quad - \pi\, a - (1-\pi)\, \mathbb{E}[Y \mid T=0]
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$$
\begin{aligned}
\mathbb{E}[Y(1) - Y(0)] &= \pi\, \mathbb{E}[Y \mid T = 1] + (1 - \pi)\, \mathbb{E}[Y(1) \mid T = 0] \\
&\quad - \pi\, \mathbb{E}[Y(0) \mid T = 1] - (1 - \pi)\, \mathbb{E}[Y \mid T = 0] \\
&\leq \pi\, \mathbb{E}[Y \mid T = 1] + (1 - \pi)\, \mathbb{E}[Y \mid T = 0] \\
&\quad - \pi\, a - (1 - \pi)\, \mathbb{E}[Y \mid T = 0] \\
&= \pi\, \mathbb{E}[Y \mid T = 1] - \pi\, a
\end{aligned}
$$

(Observational-Counterfactual Decomposition)

OTS Lower Bound 1

OTS Lower Bound 1

OTS assumption tells us that $\mathbb{E}[Y(0) \mid T = 1] \leq \mathbb{E}[Y \mid T = 1]$

OTS Lower Bound 1

OTS assumption tells us that $-\mathbb{E}[Y(0) \mid T = 1] \geq -\mathbb{E}[Y \mid T = 1]$

OTS Lower Bound 1

OTS assumption tells us that $-\mathbb{E}[Y(0) \mid T=1] \geq -\mathbb{E}[Y \mid T=1]$

$$\begin{aligned}\mathbb{E}[Y(1)-Y(0)] = \pi\,\mathbb{E}[Y \mid T=1] + (1-\pi)\,\mathbb{E}[Y(1) \mid T=0] \\ -\pi\,\mathbb{E}[Y(0) \mid T=1] - (1-\pi)\,\mathbb{E}[Y \mid T=0]\end{aligned}$$

(Observational-Counterfactual Decomposition)

OTS Lower Bound 1

OTS assumption tells us that $-\mathbb{E}[Y(0) \mid T=1] \geq -\mathbb{E}[Y \mid T=1]$

$$\begin{aligned}
\mathbb{E}[Y(1) - Y(0)] &= \pi\, \mathbb{E}[Y \mid T=1] + (1-\pi)\, \mathbb{E}[Y(1) \mid T=0] \\
&\quad - \pi\, \mathbb{E}[Y(0) \mid T=1] - (1-\pi)\, \mathbb{E}[Y \mid T=0] \\
&\geq \pi\, \mathbb{E}[Y \mid T=1] + (1-\pi)\, a \\
&\quad - \pi\, \mathbb{E}[Y \mid T=1] - (1-\pi)\, \mathbb{E}[Y \mid T=0]
\end{aligned}$$

(Observational-Counterfactual Decomposition)

OTS Lower Bound 1

OTS assumption tells us that $-\mathbb{E}[Y(0) \mid T=1] \geq -\mathbb{E}[Y \mid T=1]$

$$
\begin{aligned}
\mathbb{E}[Y(1) - Y(0)] &= \pi\, \mathbb{E}[Y \mid T=1] + (1-\pi)\, \mathbb{E}[Y(1) \mid T=0] \\
&\quad - \pi\, \mathbb{E}[Y(0) \mid T=1] - (1-\pi)\, \mathbb{E}[Y \mid T=0] \\
&\geq \pi\, \mathbb{E}[Y \mid T=1] + (1-\pi)\, a \\
&\quad - \pi\, \mathbb{E}[Y \mid T=1] - (1-\pi)\, \mathbb{E}[Y \mid T=0] \\
&= (1-\pi)\, a - (1-\pi)\, \mathbb{E}[Y \mid T=0]
\end{aligned}
$$

(Observational-Counterfactual Decomposition)

OTS Complete Bound 1 and Running Example

OTS Complete Bound 1 and Running Example

$$\mathbb{E}[Y(1) - Y(0)] < \pi\, \mathbb{E}[Y \mid T = 1] - \pi\, a$$

$$\mathbb{E}[Y(1) - Y(0)] \geq (1 - \pi)\, a - (1 - \pi)\, \mathbb{E}[Y \mid T = 0]$$

$$\text{Interval Length} = \pi\, \mathbb{E}[Y \mid T = 1] + (1 - \pi)\, \mathbb{E}[Y \mid T = 0] - a$$

OTS Complete Bound 1 and Running Example

$$\mathbb{E}[Y(1) - Y(0)] < \pi\, \mathbb{E}[Y \mid T = 1] - \pi\, a$$

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$$\text{Interval Length} = \pi\, \mathbb{E}[Y \mid T = 1] + (1 - \pi)\, \mathbb{E}[Y \mid T = 0] - a$$

Running example

Potential outcomes bounded between 0 (a) and 1 (b)

$$\pi = 0.3 \qquad \mathbb{E}[Y \mid T = 1] = .9 \qquad \mathbb{E}[Y \mid T = 0] = .2$$

OTS Complete Bound 1 and Running Example

$$\mathbb{E}[Y(1) - Y(0)] < \pi \, \mathbb{E}[Y \mid T = 1] - \pi \, a$$
$$\mathbb{E}[Y(1) - Y(0)] \geq (1 - \pi) \, a - (1 - \pi) \, \mathbb{E}[Y \mid T = 0]$$
$$\text{Interval Length} = \pi \, \mathbb{E}[Y \mid T = 1] + (1 - \pi) \, \mathbb{E}[Y \mid T = 0] - a$$

Running example

Potential outcomes bounded between 0 (a) and 1 (b)

$\pi = 0.3$ $\quad$ $\mathbb{E}[Y \mid T = 1] = .9$ $\quad$ $\mathbb{E}[Y \mid T = 0] = .2$

No-assumptions bound: $-0.17 \leq \mathbb{E}[Y(1) - Y(0)] \leq 0.83$

OTS Complete Bound 1 and Running Example

$$\mathbb{E}[Y(1) - Y(0)] < \pi\, \mathbb{E}[Y \mid T = 1] - \pi\, a$$

$$\mathbb{E}[Y(1) - Y(0)] \geq (1 - \pi)\, a - (1 - \pi)\, \mathbb{E}[Y \mid T = 0]$$

$$\text{Interval Length} = \pi\, \mathbb{E}[Y \mid T = 1] + (1 - \pi)\, \mathbb{E}[Y \mid T = 0] - a$$

Running example

Potential outcomes bounded between 0 (a) and 1 (b)

$$\pi = 0.3 \qquad \mathbb{E}[Y \mid T = 1] = .9 \qquad \mathbb{E}[Y \mid T = 0] = .2$$

No-assumptions bound: $-0.17 \leq \mathbb{E}[Y(1) - Y(0)] \leq 0.83$

OTS Bound 1: $-0.14 \leq \mathbb{E}[Y(1) - Y(0)] \leq 0.27$

Interval Length $= 0.41$

Bound that identifies the sign

OTS Upper Bound 2 Preliminaries

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OTS Assumption: $T_i = 1 \implies Y_i(1) \geq Y_i(0)\,, \quad T_i = 0 \implies Y_i(0) > Y_i(1)$

OTS Upper Bound 2 Preliminaries

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OTS Bound 1 implication we used: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 0]$

OTS Upper Bound 2 Preliminaries

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OTS Bound 1 implication we used: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 0]$

OTS Bound 2 implication we'll use: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 1]$

OTS Upper Bound 2 Preliminaries

OTS Assumption: $T_i = 1 \implies Y_i(1) \geq Y_i(0)\,, \quad T_i = 0 \implies Y_i(0) > Y_i(1)$

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OTS Bound 2 implication we'll use: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 1]$

Proof: $\mathbb{E}[Y(1) \mid T = 0] = \mathbb{E}[Y(1) \mid Y(0) > Y(1)]$

OTS Upper Bound 2 Preliminaries

OTS Assumption: $T_i = 1 \implies Y_i(1) \geq Y_i(0)\,, \quad \underline{T_i = 0 \implies Y_i(0) > Y_i(1)}$

OTS Bound 1 implication we used: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 0]$

OTS Bound 2 implication we'll use: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 1]$

Proof: $\mathbb{E}[Y(1) \mid T = 0] = \mathbb{E}[Y(1) \mid Y(0) > Y(1)]$

OTS Upper Bound 2 Preliminaries

OTS Assumption: $T_i = 1 \implies Y_i(1) \geq Y_i(0)\,, \quad T_i = 0 \implies Y_i(0) > Y_i(1)$

OTS Bound 1 implication we used: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 0]$

OTS Bound 2 implication we'll use: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 1]$

Proof:
$$\begin{aligned} \mathbb{E}[Y(1) \mid T = 0] &= \mathbb{E}[Y(1) \mid Y(0) > Y(1)] \\ &\leq \mathbb{E}[Y(1) \mid Y(0) \leq Y(1)] \end{aligned}$$

OTS Upper Bound 2 Preliminaries

OTS Assumption: $T_i = 1 \implies Y_i(1) \geq Y_i(0)\,, \quad T_i = 0 \implies Y_i(0) > Y_i(1)$

OTS Bound 1 implication we used: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 0]$

OTS Bound 2 implication we'll use: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 1]$

Proof:
$$\begin{aligned}\mathbb{E}[Y(1) \mid T = 0] &= \mathbb{E}[Y(1) \mid Y(0) > Y(1)] \\ &\leq \mathbb{E}[Y(1) \mid Y(0) \leq Y(1)] \\ &= \mathbb{E}[Y(1) \mid T = 1]\end{aligned}$$

OTS Upper Bound 2 Preliminaries

OTS Assumption: $T_i = 1 \implies Y_i(1) \geq Y_i(0)\,, \quad T_i = 0 \implies Y_i(0) > Y_i(1)$

Contrapositive: $T_i = 1 \impliedby Y_i(0) \leq Y_i(1)$

OTS Bound 1 implication we used: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 0]$

OTS Bound 2 implication we'll use: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 1]$

Proof:
$$\begin{aligned} \mathbb{E}[Y(1) \mid T = 0] &= \mathbb{E}[Y(1) \mid Y(0) > Y(1)] \\ &\leq \mathbb{E}[Y(1) \mid Y(0) \leq Y(1)] \\ &= \mathbb{E}[Y(1) \mid T = 1] \end{aligned}$$

OTS Upper Bound 2 Preliminaries

OTS Assumption: $T_i = 1 \implies Y_i(1) \geq Y_i(0)\,, \quad T_i = 0 \implies Y_i(0) > Y_i(1)$

Contrapositive: $T_i = 1 \impliedby Y_i(0) \leq Y_i(1)$

OTS Bound 1 implication we used: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 0]$

OTS Bound 2 implication we'll use: $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 1]$

Proof:
$$\begin{aligned}
\mathbb{E}[Y(1) \mid T = 0] &= \mathbb{E}[Y(1) \mid Y(0) > Y(1)] \\
&\leq \mathbb{E}[Y(1) \mid Y(0) \leq Y(1)] \\
&= \mathbb{E}[Y(1) \mid T = 1] \\
&= \mathbb{E}[Y \mid T = 1]
\end{aligned}$$

OTS Upper Bound 2

OTS assumption tells us that $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 1]$

OTS Upper Bound 2

OTS assumption tells us that $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 1]$

$$\begin{aligned}\mathbb{E}[Y(1) - Y(0)] = \pi\, \mathbb{E}[Y \mid T = 1] + (1 - \pi)\, \mathbb{E}[Y(1) \mid T = 0] \\ - \pi\, \mathbb{E}[Y(0) \mid T = 1] - (1 - \pi)\, \mathbb{E}[Y \mid T = 0]\end{aligned}$$

(Observational-Counterfactual Decomposition)

OTS Upper Bound 2

OTS assumption tells us that $\mathbb{E}[Y(1) \mid T=0] \leq \mathbb{E}[Y \mid T=1]$

$$
\begin{aligned}
\mathbb{E}[Y(1) - Y(0)] &= \pi\, \mathbb{E}[Y \mid T=1] + (1-\pi)\, \mathbb{E}[Y(1) \mid T=0] \\
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&\leq \pi\, \mathbb{E}[Y \mid T=1] + (1-\pi)\, \mathbb{E}[Y \mid T=1] \\
&\quad - \pi\, a - (1-\pi)\, \mathbb{E}[Y \mid T=0]
\end{aligned}
$$

(Observational-Counterfactual Decomposition)

OTS Upper Bound 2

OTS assumption tells us that $\mathbb{E}[Y(1) \mid T = 0] \leq \mathbb{E}[Y \mid T = 1]$

$$\begin{aligned}
\mathbb{E}[Y(1) - Y(0)] &= \pi \, \mathbb{E}[Y \mid T = 1] + (1 - \pi) \, \mathbb{E}[Y(1) \mid T = 0] \\
&\quad - \pi \, \mathbb{E}[Y(0) \mid T = 1] - (1 - \pi) \, \mathbb{E}[Y \mid T = 0] \\
&\leq \pi \, \mathbb{E}[Y \mid T = 1] + (1 - \pi) \, \mathbb{E}[Y \mid T = 1] \\
&\quad - \pi \, a - (1 - \pi) \, \mathbb{E}[Y \mid T = 0] \\
&= \mathbb{E}[Y \mid T = 1] - \pi \, a - (1 - \pi) \, \mathbb{E}[Y \mid T = 0]
\end{aligned}$$

(Observational-Counterfactual Decomposition)

Question:
Prove a new lower bound using the version of the OTS assumption that we used in the last slide.

OTS Complete Bound 2 and Running Example

OTS Complete Bound 2 and Running Example

$$\mathbb{E}[Y(1) - Y(0)] \leq \mathbb{E}[Y \mid T = 1] - \pi\, a - (1 - \pi)\, \mathbb{E}[Y \mid T = 0]$$

$$\mathbb{E}[Y(1) - Y(0)] \geq \pi\, \mathbb{E}[Y \mid T = 1] + (1 - \pi)\, a - \mathbb{E}[Y \mid T = 0]$$

$$\text{Interval Length} = (1 - \pi)\, \mathbb{E}[Y \mid T = 1] + \pi\, \mathbb{E}[Y \mid T = 0] - a$$

OTS Complete Bound 2 and Running Example

$$\mathbb{E}[Y(1) - Y(0)] \leq \mathbb{E}[Y \mid T = 1] - \pi\, a - (1 - \pi)\, \mathbb{E}[Y \mid T = 0]$$

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Running example

Potential outcomes bounded between 0 (a) and 1 (b)

$\pi = 0.3$ $\qquad \mathbb{E}[Y \mid T = 1] = .9$ $\qquad \mathbb{E}[Y \mid T = 0] = .2$

OTS Complete Bound 2 and Running Example

$$\mathbb{E}[Y(1) - Y(0)] \leq \mathbb{E}[Y \mid T = 1] - \pi\, a - (1 - \pi)\, \mathbb{E}[Y \mid T = 0]$$

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Running example

Potential outcomes bounded between 0 (a) and 1 (b)

$\pi = 0.3$ $\quad \mathbb{E}[Y \mid T = 1] = .9$ $\quad \mathbb{E}[Y \mid T = 0] = .2$

No-assumptions bound: $-0.17 \leq \mathbb{E}[Y(1) - Y(0)] \leq 0.83$

OTS Complete Bound 2 and Running Example

$$\mathbb{E}[Y(1) - Y(0)] \leq \mathbb{E}[Y \mid T = 1] - \pi\, a - (1 - \pi)\, \mathbb{E}[Y \mid T = 0]$$

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Running example

Potential outcomes bounded between 0 (a) and 1 (b)

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No-assumptions bound: $-0.17 \leq \mathbb{E}[Y(1) - Y(0)] \leq 0.83$

OTS Bound 2: $0.07 \leq \mathbb{E}[Y(1) - Y(0)] \leq 0.76$

Interval Length $= 0.69$

OTS Complete Bound 2 and Running Example

$$\mathbb{E}[Y(1) - Y(0)] \leq \mathbb{E}[Y \mid T = 1] - \pi\, a - (1 - \pi)\, \mathbb{E}[Y \mid T = 0]$$

$$\mathbb{E}[Y(1) - Y(0)] \geq \pi\, \mathbb{E}[Y \mid T = 1] + (1 - \pi)\, a - \mathbb{E}[Y \mid T = 0]$$

$$\text{Interval Length} = (1 - \pi)\, \mathbb{E}[Y \mid T = 1] + \pi\, \mathbb{E}[Y \mid T = 0] - a$$

Running example

Potential outcomes bounded between 0 (a) and 1 (b)

$\pi = 0.3$ $\qquad \mathbb{E}[Y \mid T = 1] = .9$ $\qquad \mathbb{E}[Y \mid T = 0] = .2$

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Interval Length $= 0.69$

Identified the sign of the effect!

Comparing and Mixing OTS Bounds

Potential outcomes bounded between 0 (a) and 1 (b)

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Comparing and Mixing OTS Bounds

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Interval Length $= 0.41$

Comparing and Mixing OTS Bounds

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Comparing and Mixing OTS Bounds

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Interval Length $= 0.69$

Identified the sign of the effect, but gives a 68% larger interval

Comparing and Mixing OTS Bounds

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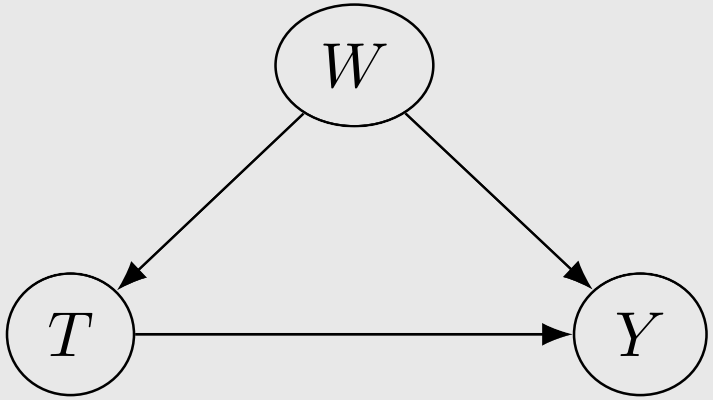
Interval Length $= 0.69$

Identified the sign of the effect, but gives a 68% larger interval

OTS Upper Bound 1 and OTS Lower Bound 2:

$$0.07 \leq \mathbb{E}[Y(1) - Y(0)] \leq 0.27 \qquad \text{Interval Length} = 0.2$$

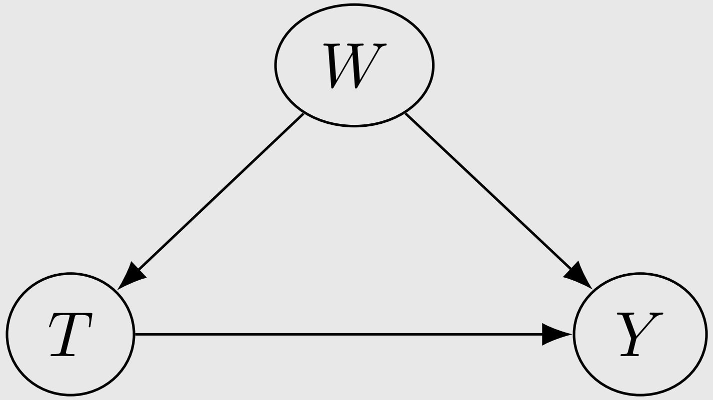
Unobserved Confounder Setting



$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_W \left[\mathbb{E}[Y \mid T = 1, W] - \mathbb{E}[Y \mid T = 0, W]\right]$$

Unobserved Confounder Setting

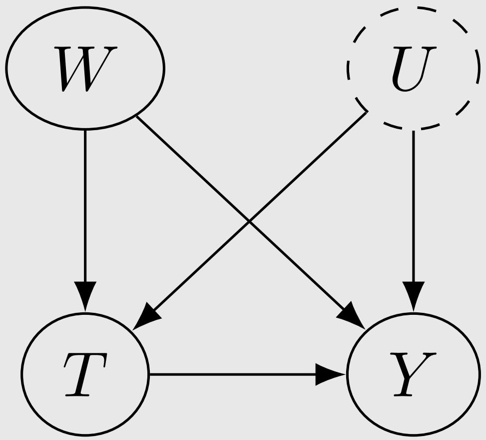
Last section, we completely threw out the unconfoundedness assumption.



$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_W \left[\mathbb{E}[Y \mid T = 1, W] - \mathbb{E}[Y \mid T = 0, W]\right]$$

Unobserved Confounder Setting

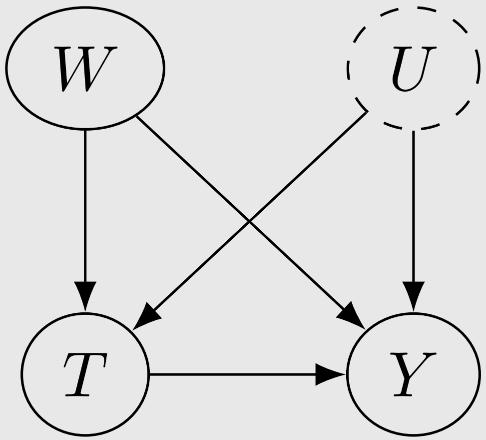
Last section, we completely threw out the unconfoundedness assumption. Now, we assume the observed W *and* the unobserved U gives us unconfoundedness



$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_{W,U}\left[\mathbb{E}[Y \mid T = 1, W, U] - \mathbb{E}[Y \mid T = 0, W, U]\right]$$

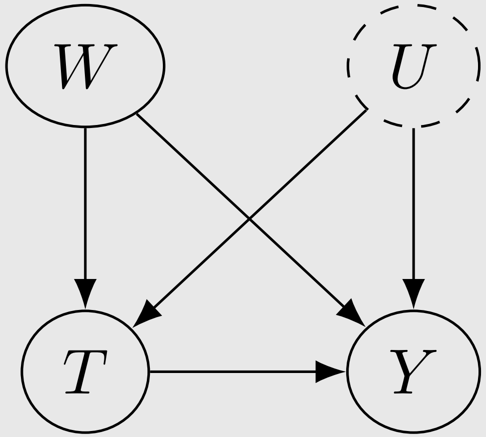
Unobserved Confounder Setting

Last section, we completely threw out the unconfoundedness assumption. Now, we assume the observed W *and* the unobserved U gives us unconfoundedness

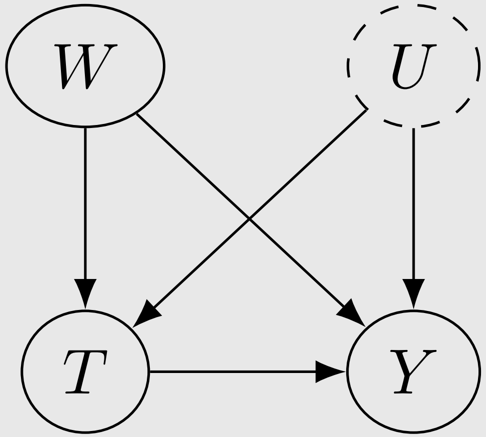


$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_{W,U}\left[\mathbb{E}[Y \mid T=1, W, U] - \mathbb{E}[Y \mid T=0, W, U]\right]$$
$$\stackrel{?}{\approx} \mathbb{E}_{W}\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right]$$

Linear SCM

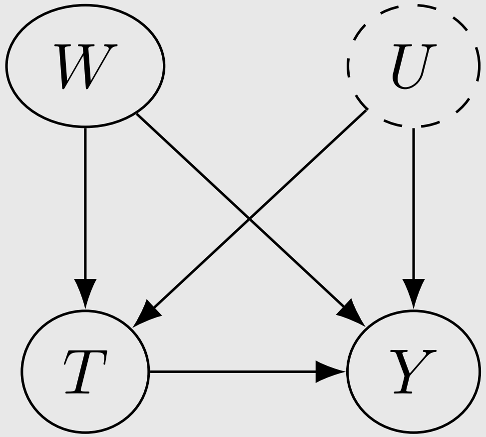


Linear SCM



$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

Linear SCM

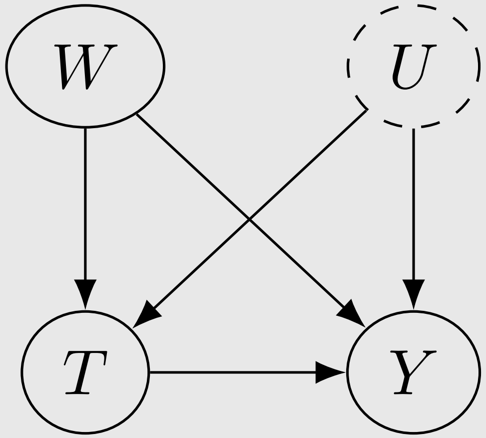


$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \underline{\delta} T$$

Goal: recover δ

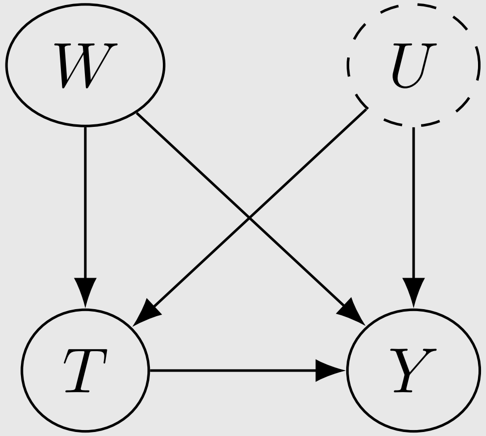
Bias in Simple Linear Setting

$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$



Bias in Simple Linear Setting

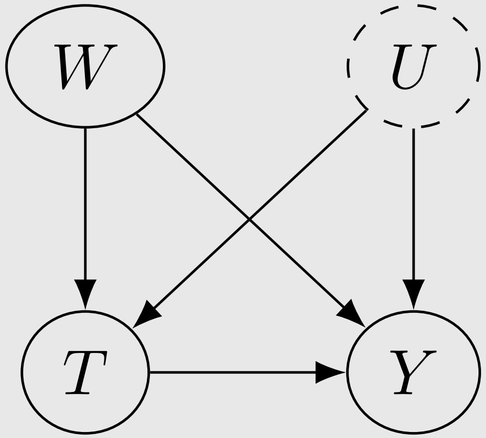
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$



$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_{W,U}\left[\mathbb{E}[Y \mid T=1, W, U] - \mathbb{E}[Y \mid T=0, W, U]\right] = \delta$$

Bias in Simple Linear Setting

$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

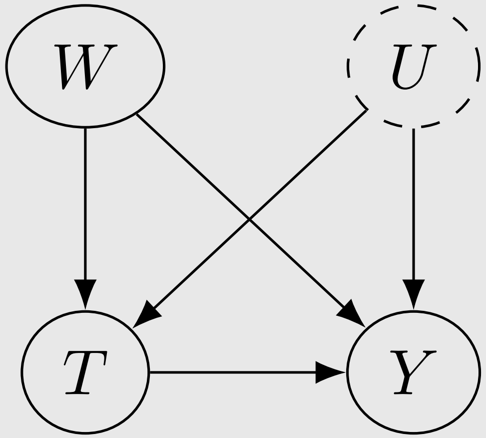


$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_{W,U}\left[\mathbb{E}[Y \mid T=1, W, U] - \mathbb{E}[Y \mid T=0, W, U]\right] = \delta$$

$$\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] \stackrel{?}{=}$$

Bias in Simple Linear Setting

$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

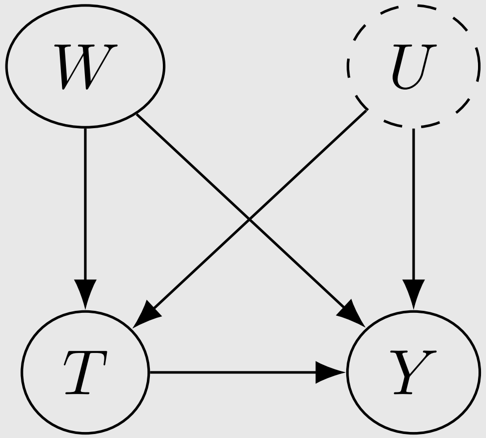


$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_{W,U}\left[\mathbb{E}[Y \mid T=1, W, U] - \mathbb{E}[Y \mid T=0, W, U]\right] = \delta$$

$$\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] = \delta + \frac{\beta_u}{\alpha_u}$$

Bias in Simple Linear Setting

$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$



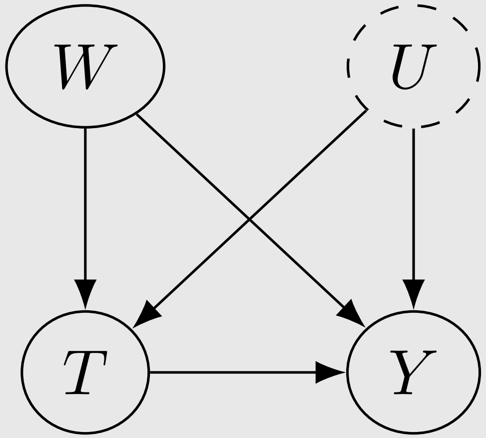
Proof coming after next slide

$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_{W,U}\left[\mathbb{E}[Y \mid T=1, W, U] - \mathbb{E}[Y \mid T=0, W, U]\right] = \delta$$

$$\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] = \delta + \frac{\beta_u}{\alpha_u}$$

Bias in Simple Linear Setting

$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$



Proof coming after next slide

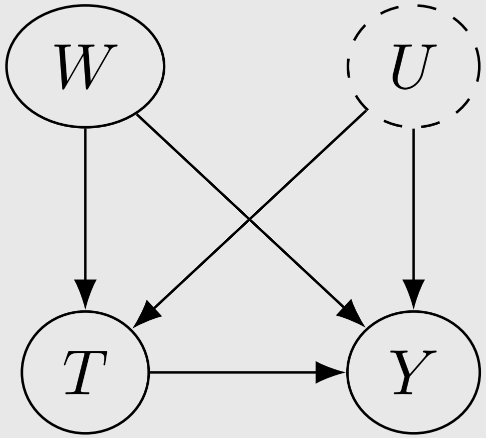
$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_{W,U}\left[\mathbb{E}[Y \mid T=1, W, U] - \mathbb{E}[Y \mid T=0, W, U]\right] = \delta$$

$$\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] = \delta + \frac{\beta_u}{\alpha_u}$$

Bias of $\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] = \delta + \frac{\beta_u}{\alpha_u} - \delta$

Bias in Simple Linear Setting

$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$



Proof coming after next slide

$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_{W,U}\left[\mathbb{E}[Y \mid T=1, W, U] - \mathbb{E}[Y \mid T=0, W, U]\right] = \delta$$

$$\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] = \delta + \frac{\beta_u}{\alpha_u}$$

$$\text{Bias of } \mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] = \delta + \frac{\beta_u}{\alpha_u} - \delta = \frac{\beta_u}{\alpha_u}$$

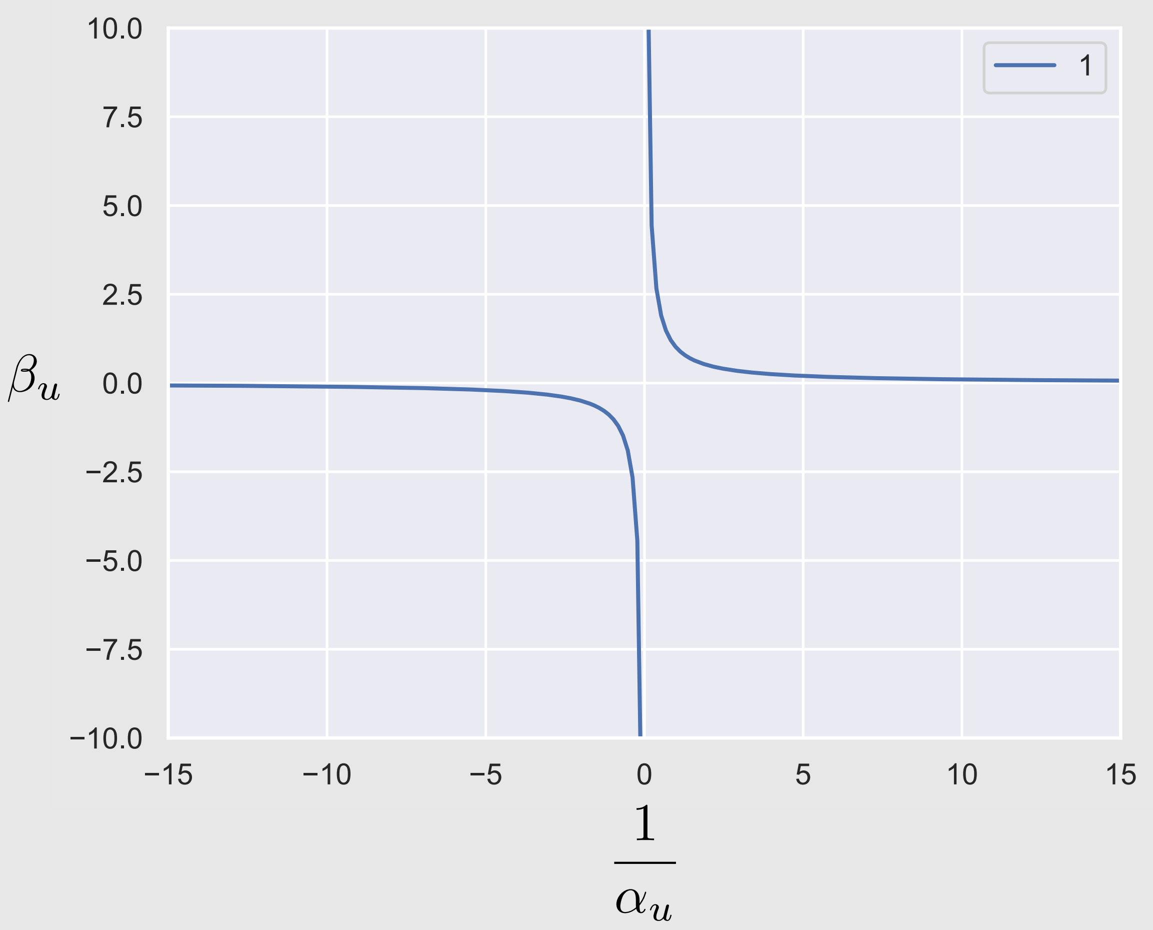
Contour Plots for Sensitivity to Confounding

Contour Plots for Sensitivity to Confounding

Bias of $\mathbb{E}_W \left[\mathbb{E}[Y \mid T = 1, W] - \mathbb{E}[Y \mid T = 0, W] \right]$

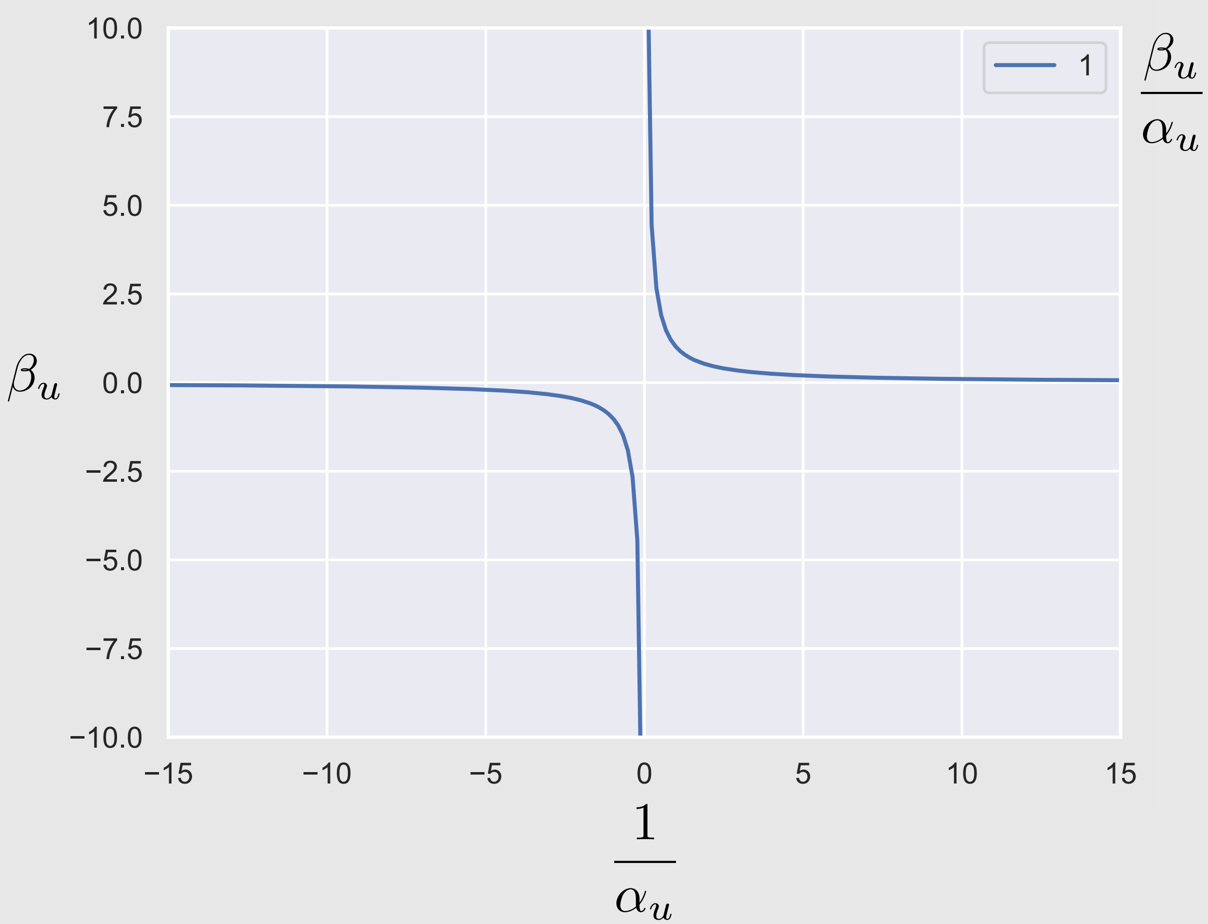
Contour Plots for Sensitivity to Confounding

Bias of $\mathbb{E}_W \left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right]$



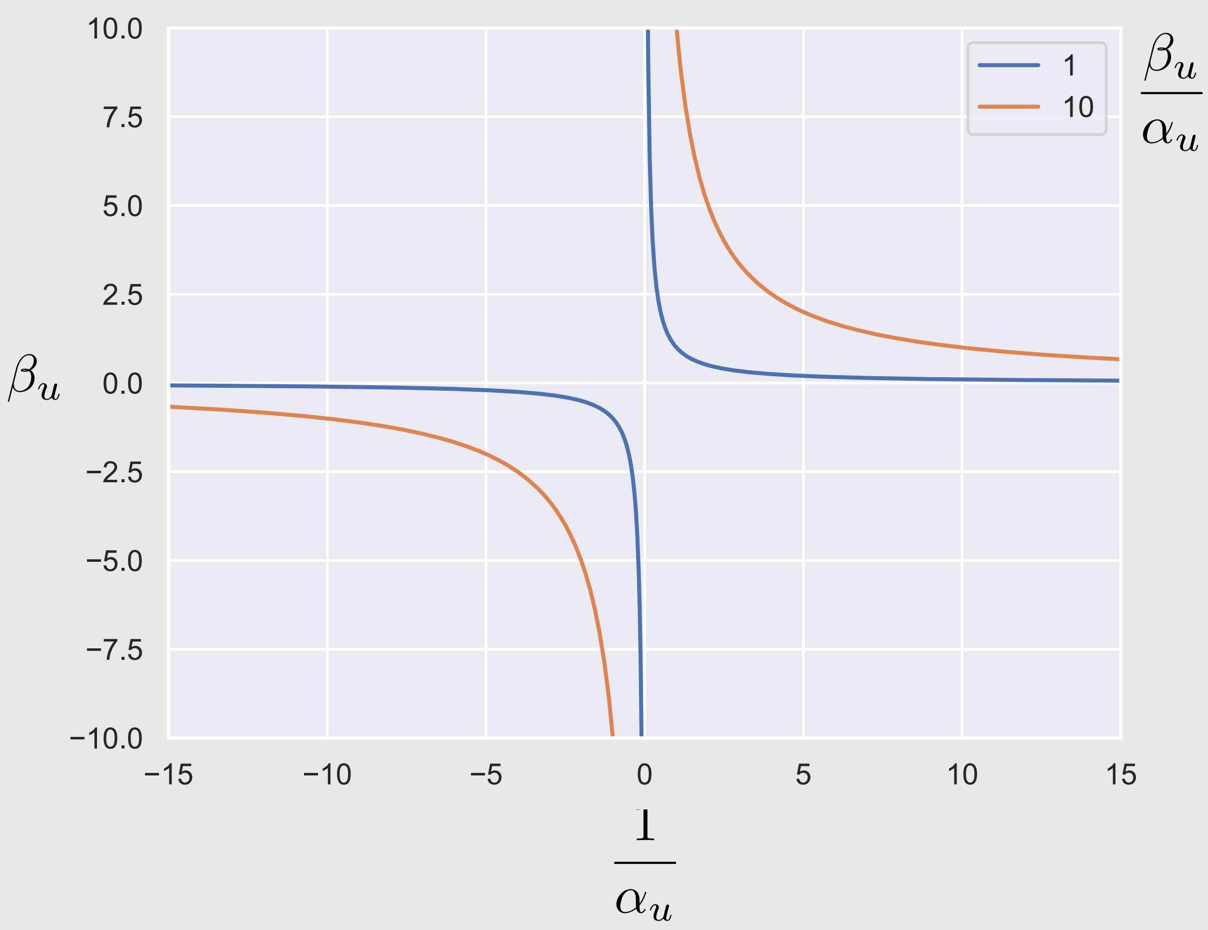
Contour Plots for Sensitivity to Confounding

Bias of $\mathbb{E}_W \left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W] \right]$



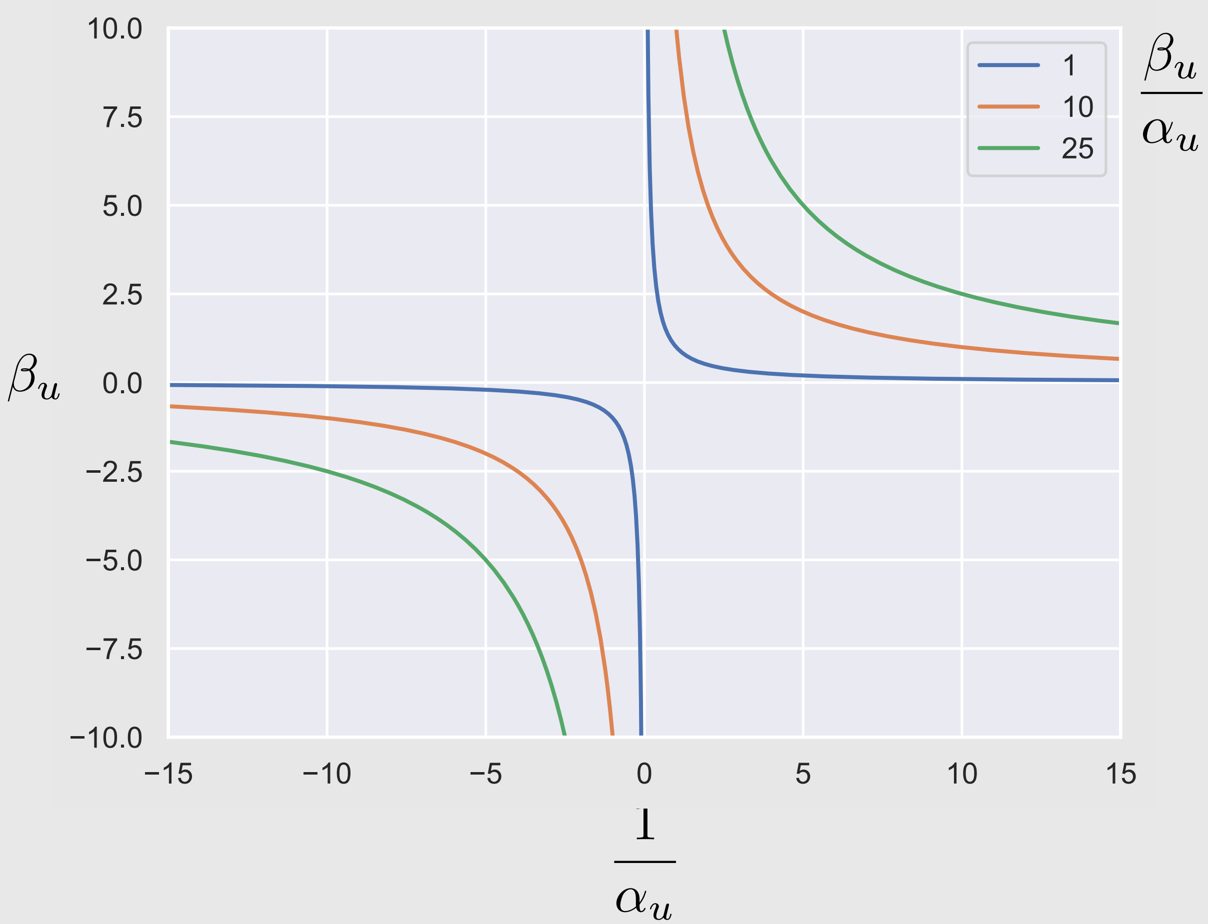
Contour Plots for Sensitivity to Confounding

Bias of $\mathbb{E}_W \left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W] \right]$



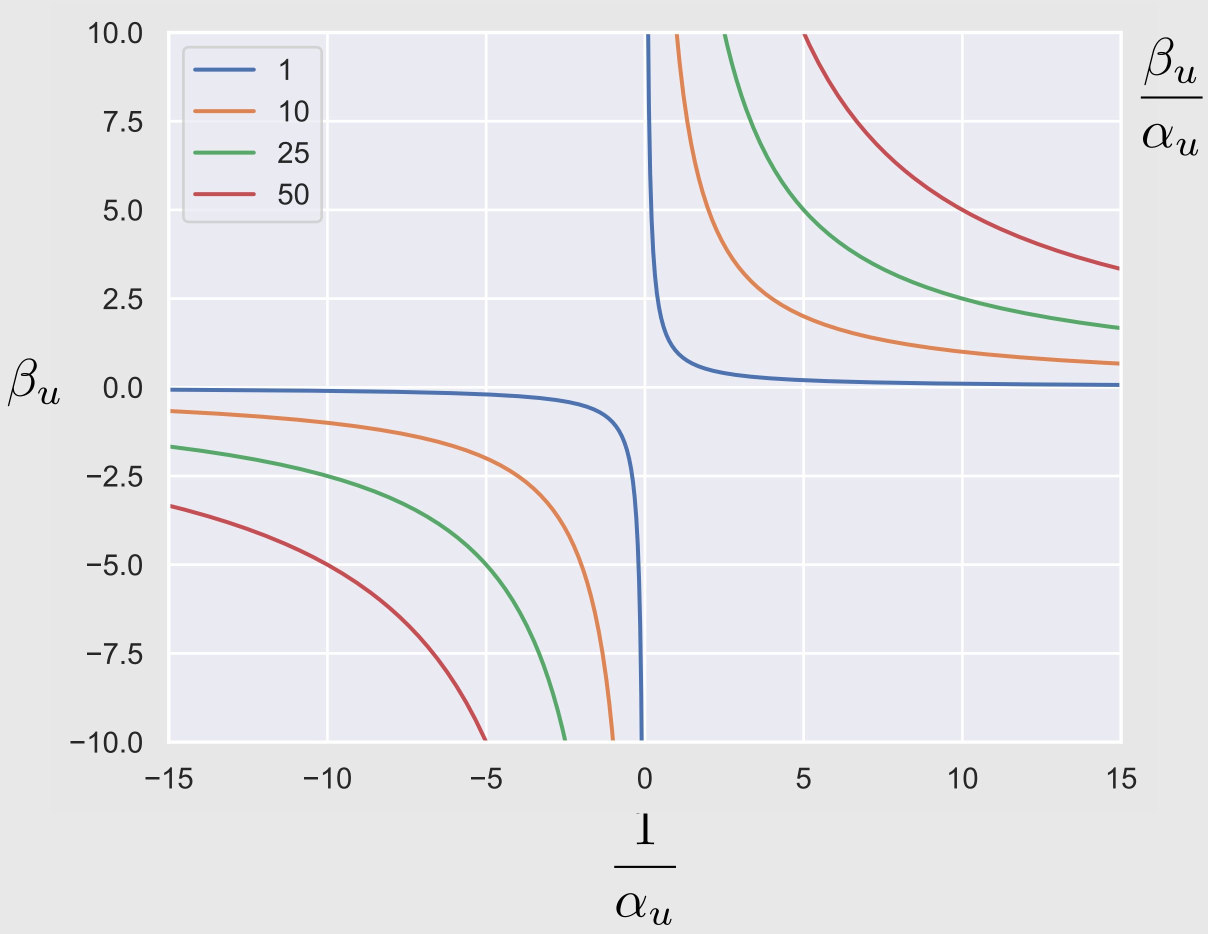
Contour Plots for Sensitivity to Confounding

Bias of $\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right]$



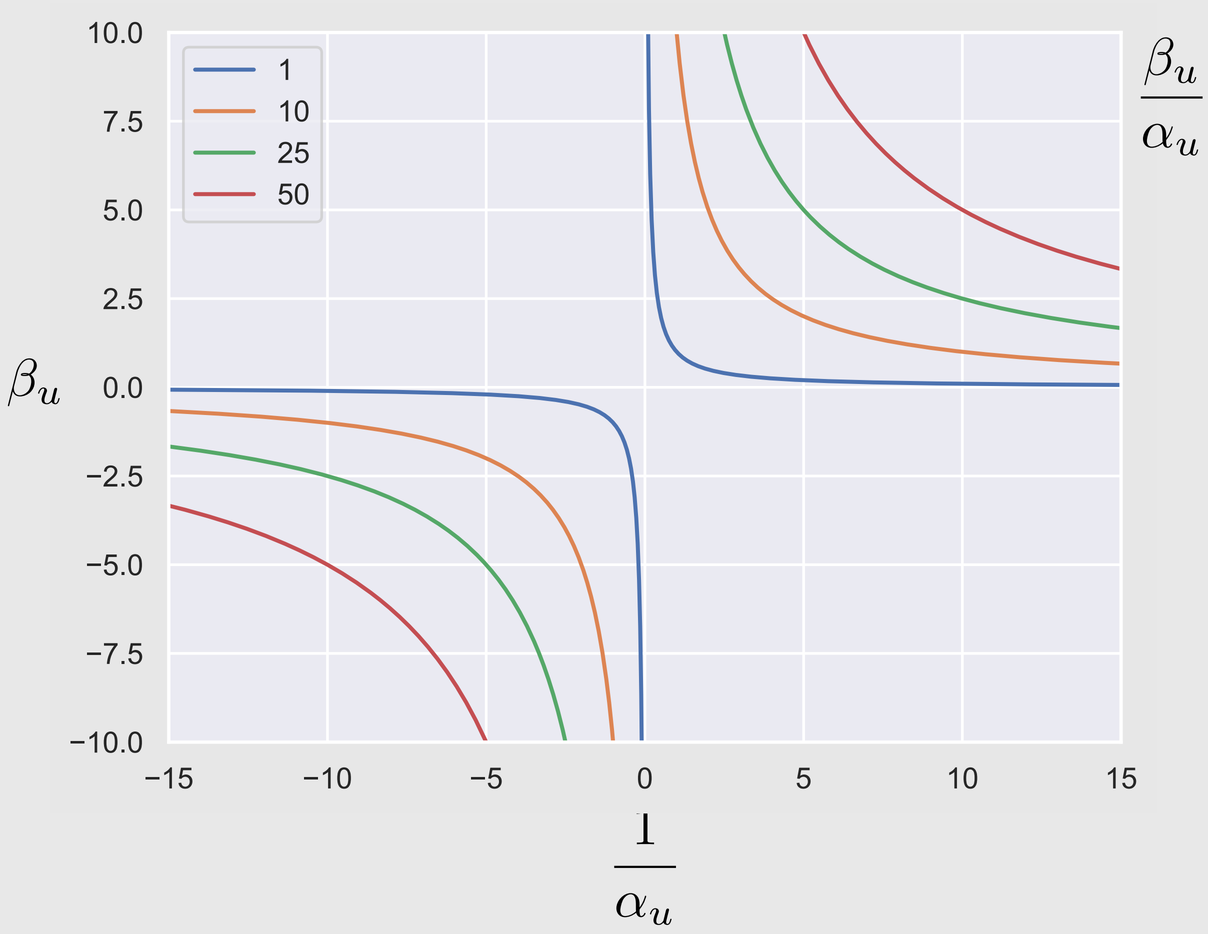
Contour Plots for Sensitivity to Confounding

Bias of $\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right]$

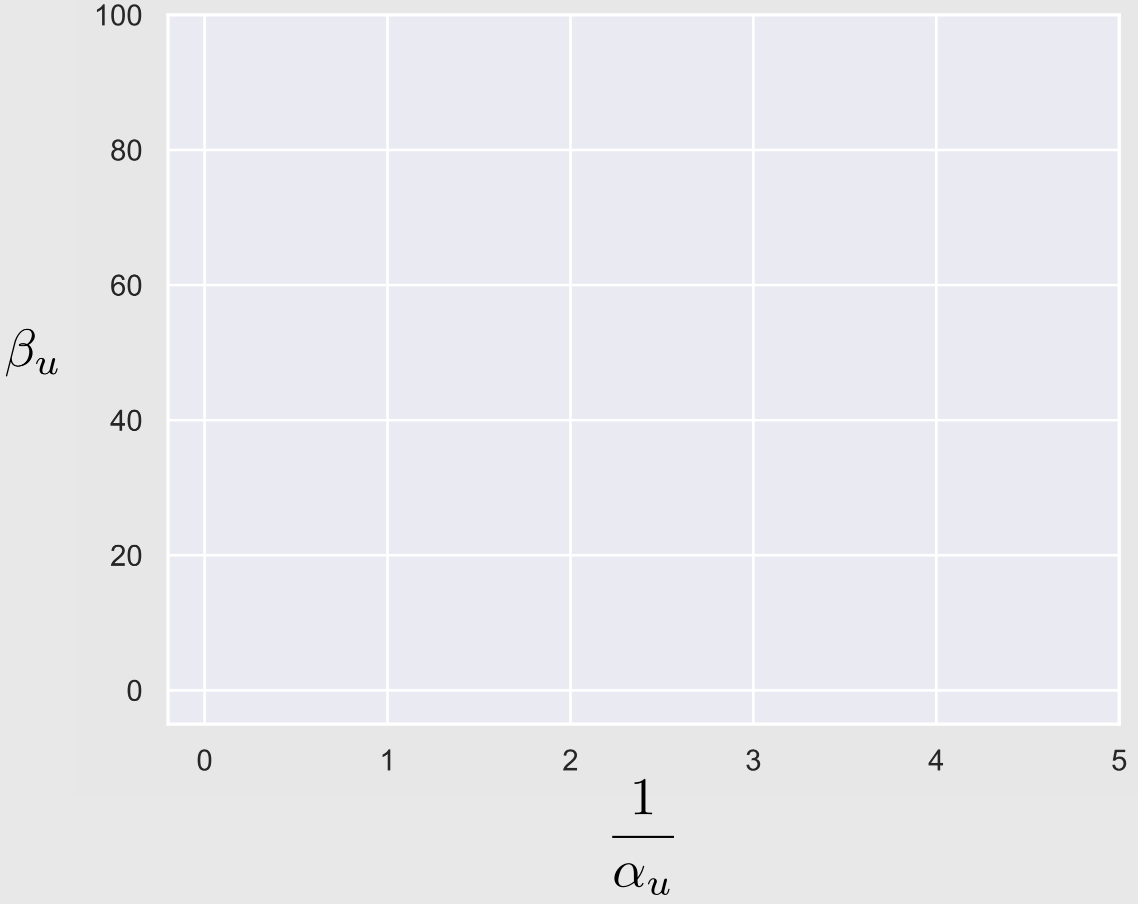


Contour Plots for Sensitivity to Confounding

Bias of $\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right]$



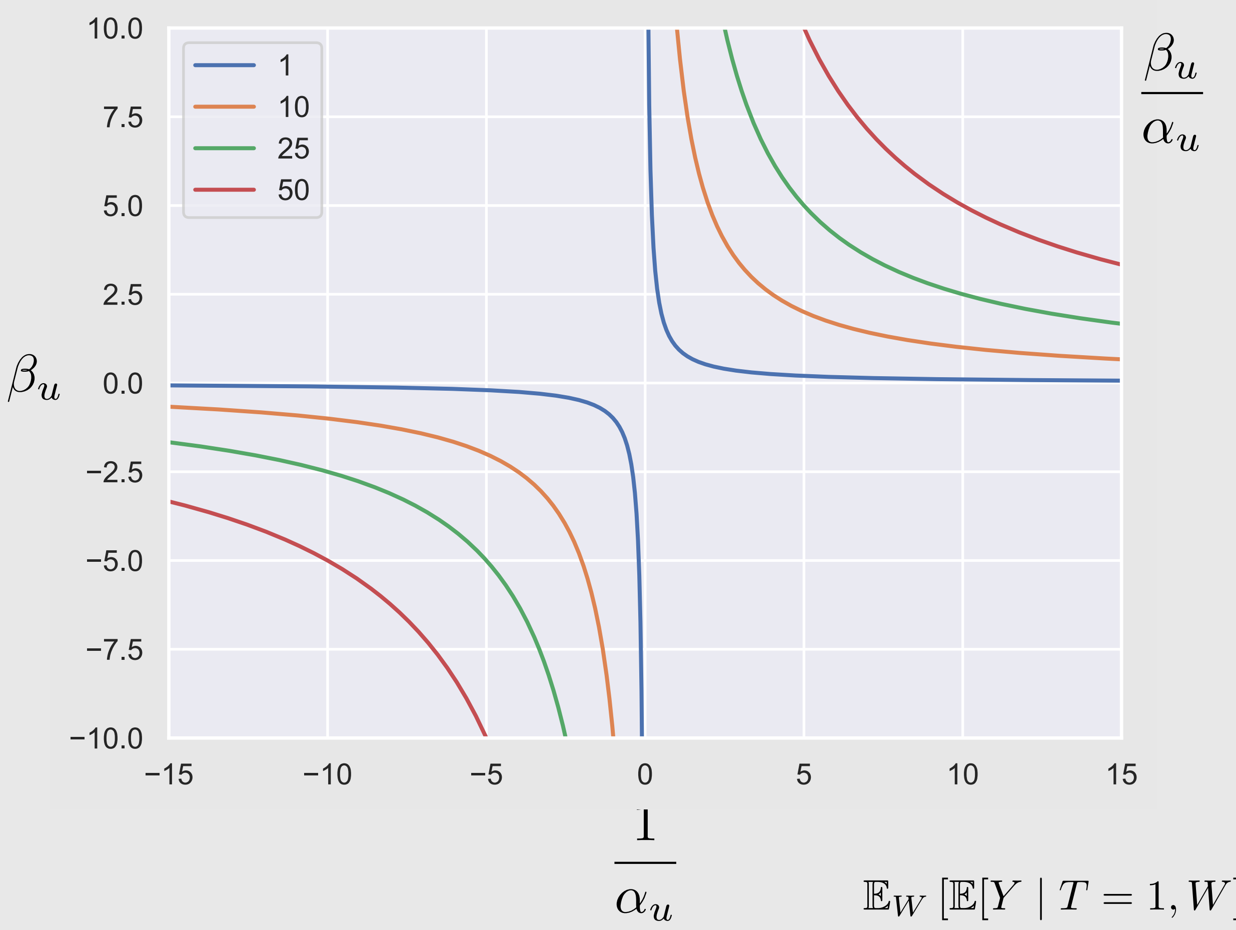
$\delta = \mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] - \frac{\beta_u}{\alpha_u}$

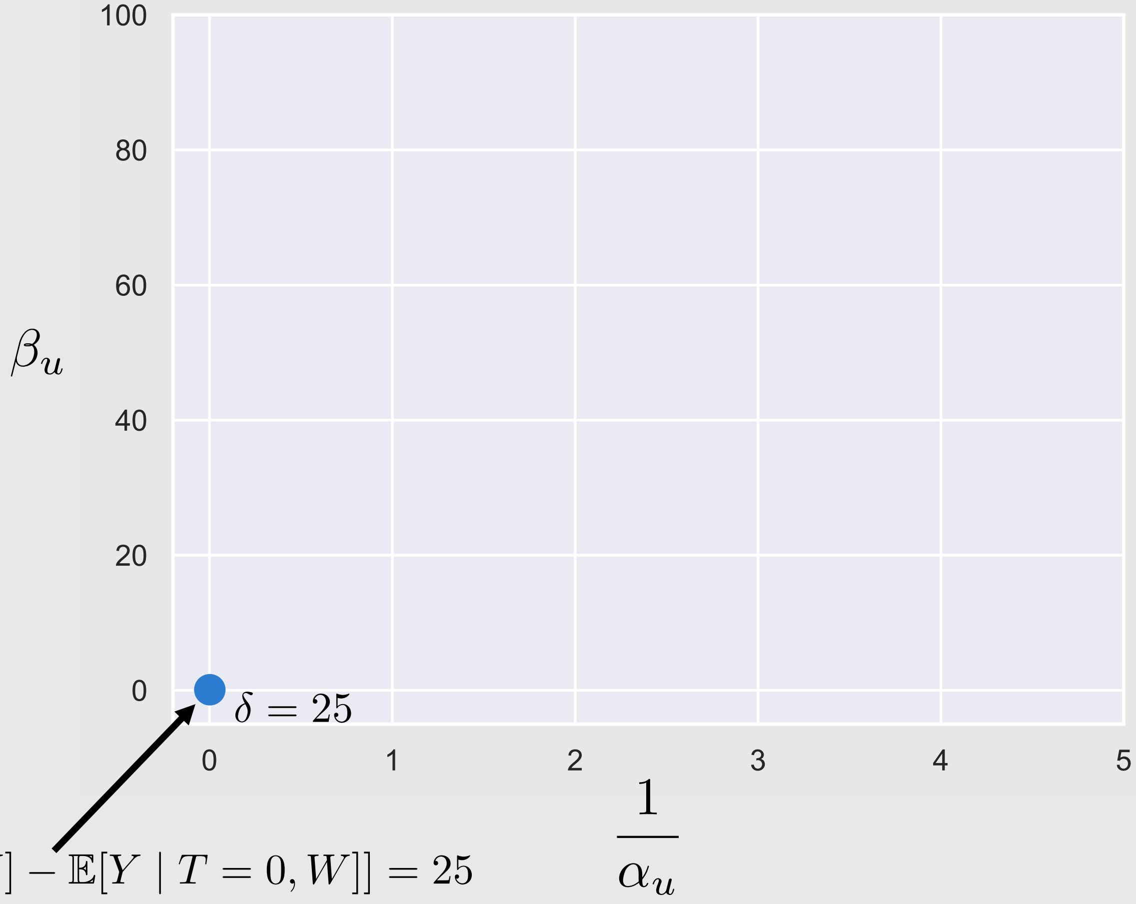


Contour Plots for Sensitivity to Confounding

Bias of $\mathbb{E}_W \left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W] \right]$

$\delta = \mathbb{E}_W \left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W] \right] - \frac{\beta_u}{\alpha_u}$

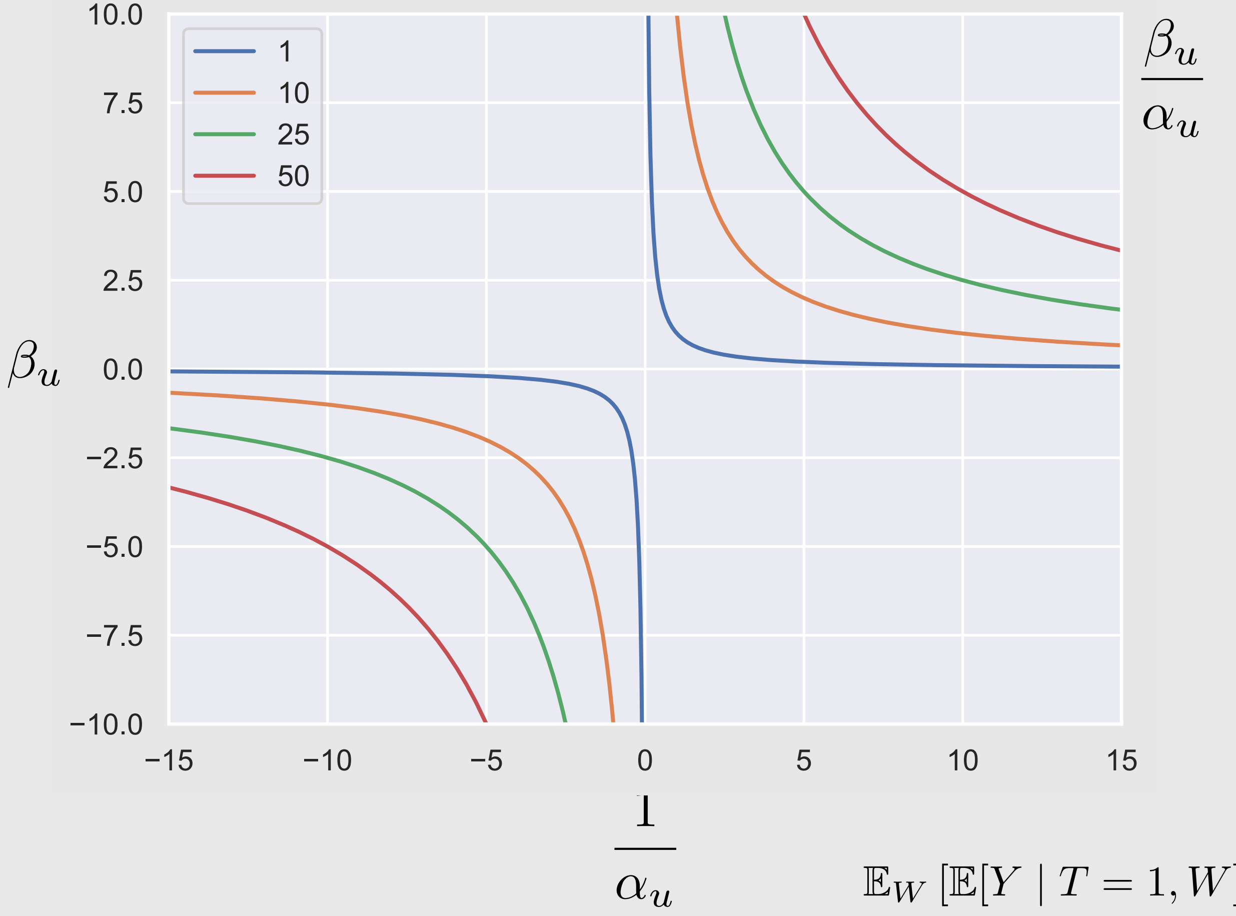




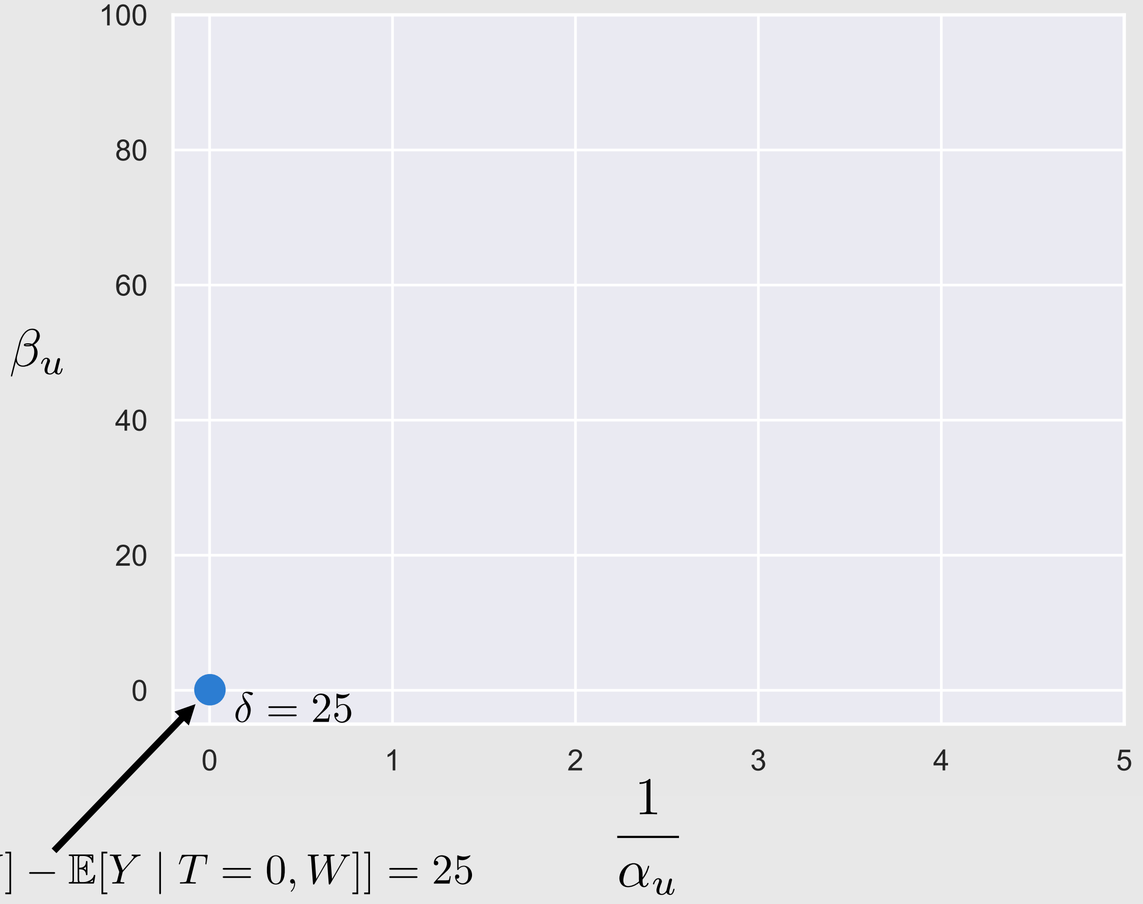
$\mathbb{E}_W \left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W] \right] = 25$

Contour Plots for Sensitivity to Confounding

Bias of $\mathbb{E}_W \left[\mathbb{E}[Y \mid T = 1, W] - \mathbb{E}[Y \mid T = 0, W] \right]$



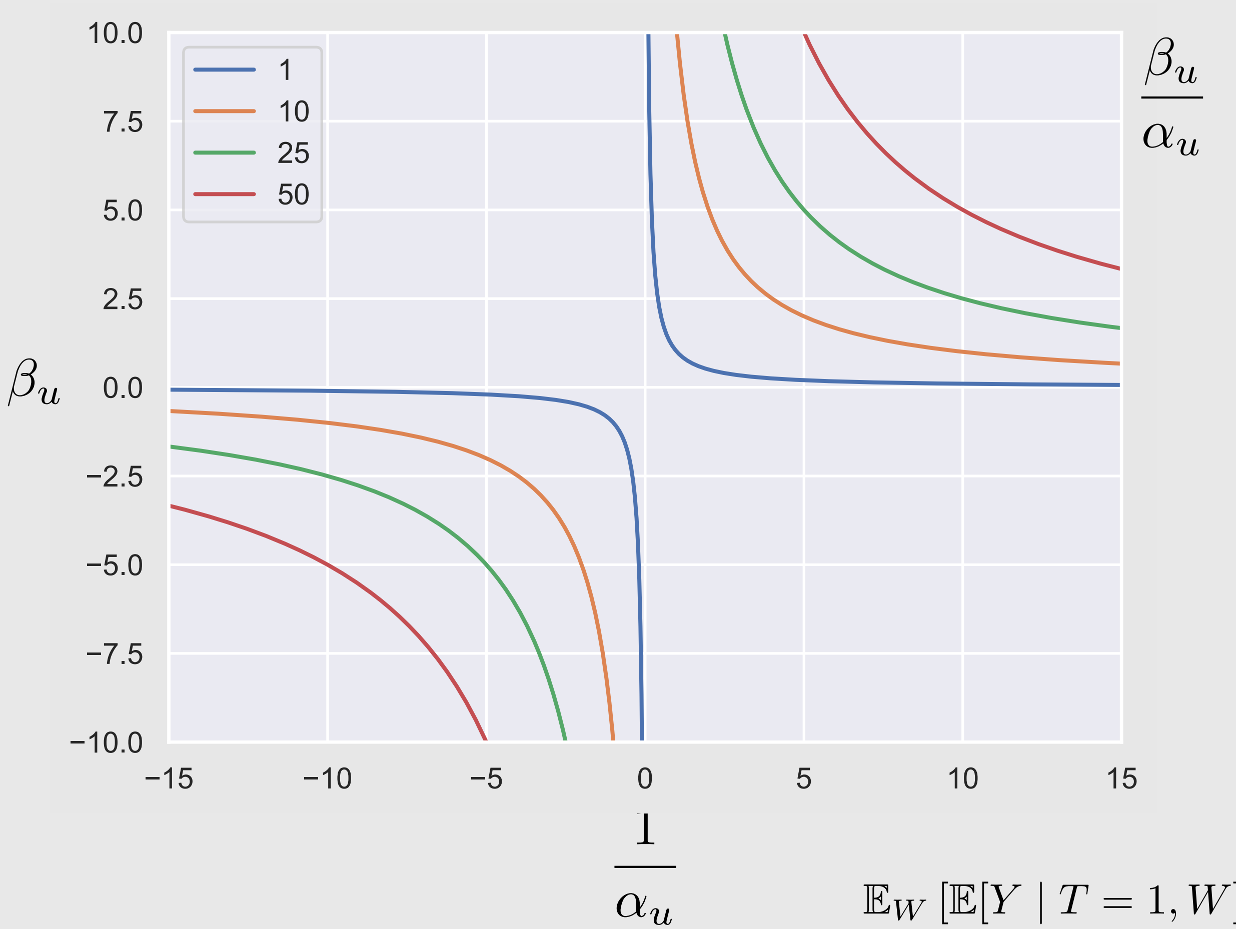
$\delta = 25 - \frac{\beta_u}{\alpha_u}$



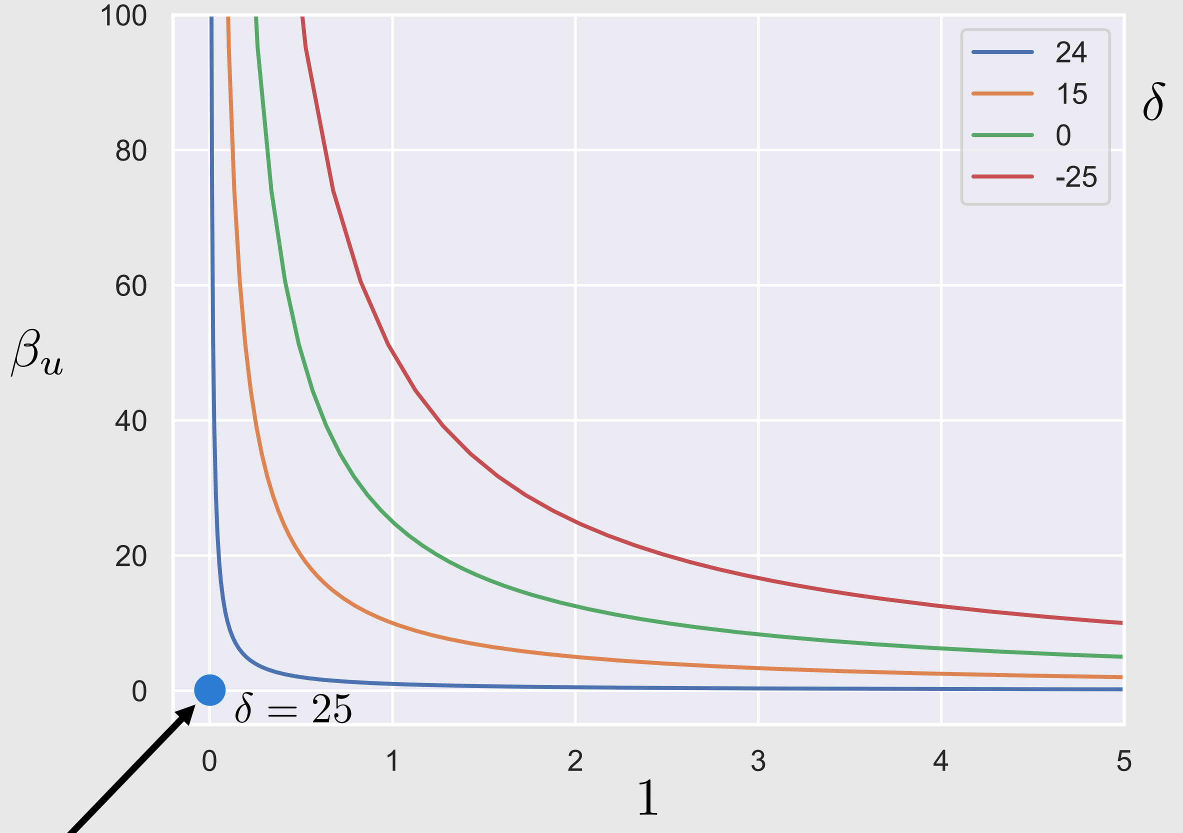
$\mathbb{E}_W \left[\mathbb{E}[Y \mid T = 1, W] - \mathbb{E}[Y \mid T = 0, W] \right] = 25$

Contour Plots for Sensitivity to Confounding

Bias of $\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right]$



$$\delta = 25 - \frac{\beta_u}{\alpha_u}$$



$\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] = 25$

Bias in Simple Linear Setting Proof: Outline

Bias in Simple Linear Setting Proof: Outline

Assumed SCM:

$$
\begin{aligned}
T &:= \alpha_w W + \alpha_u U \\
Y &:= \beta_w W + \beta_u U + \delta T
\end{aligned}
$$

Bias in Simple Linear Setting Proof: Outline

Assumed SCM:
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

Result:

The confounding bias of adjusting for just W (and not U) is $\frac{\beta_u}{\alpha_u}$. Formally,

$$\mathbb{E}_W \left[\mathbb{E}[Y \mid T = 1, W] - \mathbb{E}[Y \mid T = 0, W]\right]$$
$$- \mathbb{E}_{W,U} \left[\mathbb{E}[Y \mid T = 1, W, U] - \mathbb{E}[Y \mid T = 0, W, U]\right] = \frac{\beta_u}{\alpha_u}$$

Bias in Simple Linear Setting Proof: Outline

Assumed SCM:
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

Result:

The confounding bias of adjusting for just W (and not U) is $\frac{\beta_u}{\alpha_u}$. Formally,

$$\mathbb{E}_W \left[\mathbb{E}[Y \mid T = 1, W] - \mathbb{E}[Y \mid T = 0, W]\right]$$
$$- \mathbb{E}_{W,U} \left[\mathbb{E}[Y \mid T = 1, W, U] - \mathbb{E}[Y \mid T = 0, W, U]\right] = \frac{\beta_u}{\alpha_u}$$

Proof Outline:

1. Get a closed-form expression for $\mathbb{E}_W \left[\mathbb{E}[Y \mid T = t, W]\right]$ in terms of α_w, α_u, β_w, and β_u.

Bias in Simple Linear Setting Proof: Outline

Assumed SCM:
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

Result:

The confounding bias of adjusting for just W (and not U) is $\frac{\beta_u}{\alpha_u}$. Formally,

$$\mathbb{E}_W \left[\mathbb{E}[Y \mid T = 1, W] - \mathbb{E}[Y \mid T = 0, W]\right]$$
$$- \mathbb{E}_{W,U} \left[\mathbb{E}[Y \mid T = 1, W, U] - \mathbb{E}[Y \mid T = 0, W, U]\right] = \frac{\beta_u}{\alpha_u}$$

Proof Outline:

1. Get a closed-form expression for $\mathbb{E}_W \left[\mathbb{E}[Y \mid T = t, W]\right]$ in terms of α_w, α_u, β_w, and β_u.
2. Use step 1 to get a closed-form expression for the difference
$$\mathbb{E}_W \left[\mathbb{E}[Y \mid T = 1, W] - \mathbb{E}[Y \mid T = 0, W]\right]$$

Bias in Simple Linear Setting Proof: Outline

Assumed SCM:
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

Result:

The confounding bias of adjusting for just W (and not U) is $\frac{\beta_u}{\alpha_u}$. Formally,

$$\mathbb{E}_W \left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W] \right]$$
$$- \mathbb{E}_{W,U} \left[\mathbb{E}[Y \mid T=1, W, U] - \mathbb{E}[Y \mid T=0, W, U] \right] = \frac{\beta_u}{\alpha_u}$$

Proof Outline:

1. Get a closed-form expression for $\mathbb{E}_W \left[\mathbb{E}[Y \mid T=t, W] \right]$ in terms of α_w, α_u, β_w, and β_u.
2. Use step 1 to get a closed-form expression for the difference
$$\mathbb{E}_W \left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W] \right]$$
3. Subtract off $\mathbb{E}_{W,U} \left[\mathbb{E}[Y \mid T=1, W, U] - \mathbb{E}[Y \mid T=0, W, U] \right] = \delta$

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM:
$$\begin{aligned} T &:= \alpha_w W + \alpha_u U \\ Y &:= \beta_w W + \beta_u U + \delta T \end{aligned}$$

Get a closed-form expression for $\mathbb{E}_W \left[\mathbb{E}[Y \mid T = t, W] \right]$ in terms of α_w, α_u, β_w, and β_u.

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM:
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

$$\mathbb{E}_W \left[\mathbb{E}[Y \mid T = t, W] \right]$$

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM:
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

$$\mathbb{E}_W \left[\mathbb{E}[Y \mid T = t, W] \right]$$

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM:
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

$$\mathbb{E}_W\left[\mathbb{E}[Y \mid T = t, W]\right] = \mathbb{E}_W\left[\mathbb{E}[\beta_w W + \beta_u U + \delta T \mid T = t, W]\right]$$

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM:
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

$$\begin{aligned}\mathbb{E}_W\left[\mathbb{E}[Y \mid T=t, W]\right] &= \mathbb{E}_W\left[\mathbb{E}[\beta_w W + \beta_u U + \delta T \mid T=t, W]\right] \\ &= \mathbb{E}_W\left[\beta_w W + \beta_u \mathbb{E}[U \mid T=t, W] + \delta t\right]\end{aligned}$$

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM:
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

$$\begin{aligned}\mathbb{E}_W \left[\mathbb{E}[Y \mid T = t, W]\right] &= \mathbb{E}_W \left[\mathbb{E}[\beta_w W + \beta_u U + \delta T \mid T = t, W]\right] \\ &= \mathbb{E}_W \left[\beta_w W + \beta_u \mathbb{E}[U \mid T = t, W] + \delta t\right]\end{aligned}$$

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM:

$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

$$U = \frac{T - \alpha_w W}{\alpha_u}$$

$$\begin{aligned}\mathbb{E}_W\left[\mathbb{E}[Y \mid T = t, W]\right] &= \mathbb{E}_W\left[\mathbb{E}[\beta_w W + \beta_u U + \delta T \mid T = t, W]\right] \\ &= \mathbb{E}_W\left[\beta_w W + \beta_u \mathbb{E}[U \mid T = t, W] + \delta t\right]\end{aligned}$$

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM:

$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

$$U = \frac{T - \alpha_w W}{\alpha_u}$$

$$\begin{aligned}
\mathbb{E}_W\left[\mathbb{E}[Y \mid T = t, W]\right] &= \mathbb{E}_W\left[\mathbb{E}[\beta_w W + \beta_u U + \delta T \mid T = t, W]\right] \\
&= \mathbb{E}_W\left[\beta_w W + \beta_u \mathbb{E}[U \mid T = t, W] + \delta t\right] \\
&= \mathbb{E}_W\left[\beta_w W + \beta_u \left(\frac{t - \alpha_w W}{\alpha_u}\right) + \delta t\right]
\end{aligned}$$

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM: $$T := \alpha_w W + \alpha_u U$$ $$Y := \beta_w W + \beta_u U + \delta T$$

$$U = \frac{T - \alpha_w W}{\alpha_u}$$

$$\begin{aligned}
\mathbb{E}_W\left[\mathbb{E}[Y \mid T = t, W]\right] &= \mathbb{E}_W\left[\mathbb{E}[\beta_w W + \beta_u U + \delta T \mid T = t, W]\right] \\
&= \mathbb{E}_W\left[\beta_w W + \beta_u \mathbb{E}[U \mid T = t, W] + \delta t\right] \\
&= \mathbb{E}_W\left[\beta_w W + \beta_u \left(\frac{t - \alpha_w W}{\alpha_u}\right) + \delta t\right] \\
&= \mathbb{E}_W\left[\beta_w W + \frac{\beta_u}{\alpha_u} t - \frac{\beta_u \alpha_w}{\alpha_u} W + \delta t\right]
\end{aligned}$$

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM: $$T := \alpha_w W + \alpha_u U$$ $$Y := \beta_w W + \beta_u U + \delta T$$ $$U = \frac{T - \alpha_w W}{\alpha_u}$$

$$\begin{aligned}
\mathbb{E}_W\left[\mathbb{E}[Y \mid T = t, W]\right] &= \mathbb{E}_W\left[\mathbb{E}[\beta_w W + \beta_u U + \delta T \mid T = t, W]\right] \\
&= \mathbb{E}_W\left[\beta_w W + \beta_u \mathbb{E}[U \mid T = t, W] + \delta t\right] \\
&= \mathbb{E}_W\left[\beta_w W + \beta_u \left(\frac{t - \alpha_w W}{\alpha_u}\right) + \delta t\right] \\
&= \mathbb{E}_W\left[\beta_w W + \frac{\beta_u}{\alpha_u} t - \frac{\beta_u \alpha_w}{\alpha_u} W + \delta t\right] \\
&= \beta_w \mathbb{E}[W] + \frac{\beta_u}{\alpha_u} t - \frac{\beta_u \alpha_w}{\alpha_u} \mathbb{E}[W] + \delta t
\end{aligned}$$

Bias in Simple Linear Setting Proof: Step 1

Assumed SCM: $\begin{aligned} T &:= \alpha_w W + \alpha_u U \\ Y &:= \beta_w W + \beta_u U + \delta T \end{aligned}$ $\quad U = \dfrac{T - \alpha_w W}{\alpha_u}$

$$
\begin{aligned}
\mathbb{E}_W\left[\mathbb{E}[Y \mid T = t, W]\right] &= \mathbb{E}_W\left[\mathbb{E}[\beta_w W + \beta_u U + \delta T \mid T = t, W]\right] \\
&= \mathbb{E}_W\left[\beta_w W + \beta_u \mathbb{E}[U \mid T = t, W] + \delta t\right] \\
&= \mathbb{E}_W\left[\beta_w W + \beta_u \left(\frac{t - \alpha_w W}{\alpha_u}\right) + \delta t\right] \\
&= \mathbb{E}_W\left[\beta_w W + \frac{\beta_u}{\alpha_u} t - \frac{\beta_u \alpha_w}{\alpha_u} W + \delta t\right] \\
&= \beta_w \mathbb{E}[W] + \frac{\beta_u}{\alpha_u} t - \frac{\beta_u \alpha_w}{\alpha_u} \mathbb{E}[W] + \delta t \\
&= \left(\delta + \frac{\beta_u}{\alpha_u}\right) t + \left(\beta_w - \frac{\beta_u \alpha_w}{\alpha_u}\right) \mathbb{E}[W]
\end{aligned}
$$

Bias in Simple Linear Setting Proof: Step 2

Step 1: $$\mathbb{E}_W\left[\mathbb{E}[Y \mid T=t, W]\right] = \left(\delta + \frac{\beta_u}{\alpha_u}\right) t + \left(\beta_w - \frac{\beta_u \alpha_w}{\alpha_u}\right) \mathbb{E}[W]$$

Bias in Simple Linear Setting Proof: Step 2

Step 1: $\mathbb{E}_W \left[\mathbb{E}[Y \mid T = t, W]\right] = \left(\delta + \frac{\beta_u}{\alpha_u}\right) t + \left(\beta_w - \frac{\beta_u \alpha_w}{\alpha_u}\right) \mathbb{E}[W]$

$$\mathbb{E}_W \left[\mathbb{E}[Y \mid T = 1, W] - \mathbb{E}[Y \mid T = 0, W]\right]$$

Bias in Simple Linear Setting Proof: Step 2

Step 1: $\mathbb{E}_W\left[\mathbb{E}[Y \mid T=t, W]\right] = \left(\delta + \frac{\beta_u}{\alpha_u}\right) t + \left(\beta_w - \frac{\beta_u \alpha_w}{\alpha_u}\right) \mathbb{E}[W]$

$$\begin{aligned}\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] = &\left(\delta + \frac{\beta_u}{\alpha_u}\right)(1) + \left(\beta_w - \frac{\beta_u \alpha_w}{\alpha_u}\right) \mathbb{E}[W] \\ &- \left[\left(\delta + \frac{\beta_u}{\alpha_u}\right)(0) + \left(\beta_w - \frac{\beta_u \alpha_w}{\alpha_u}\right) \mathbb{E}[W]\right]\end{aligned}$$

Bias in Simple Linear Setting Proof: Step 2

$$\text{Step 1:} \quad \mathbb{E}_W\left[\mathbb{E}[Y \mid T=t, W]\right] = \left(\delta + \frac{\beta_u}{\alpha_u}\right) t + \left(\beta_w - \frac{\beta_u \alpha_w}{\alpha_u}\right) \mathbb{E}[W]$$

$$\begin{aligned}
\mathbb{E}_W\left[\mathbb{E}[Y \mid T=1, W] - \mathbb{E}[Y \mid T=0, W]\right] &= \left(\delta + \frac{\beta_u}{\alpha_u}\right)(1) + \left(\beta_w - \frac{\beta_u \alpha_w}{\alpha_u}\right) \mathbb{E}[W] \\
&\quad - \left[\left(\delta + \frac{\beta_u}{\alpha_u}\right)(0) + \left(\beta_w - \frac{\beta_u \alpha_w}{\alpha_u}\right) \mathbb{E}[W]\right] \\
&= \delta + \frac{\beta_u}{\alpha_u}
\end{aligned}$$

Bias in Simple Linear Setting Proof: Step 3

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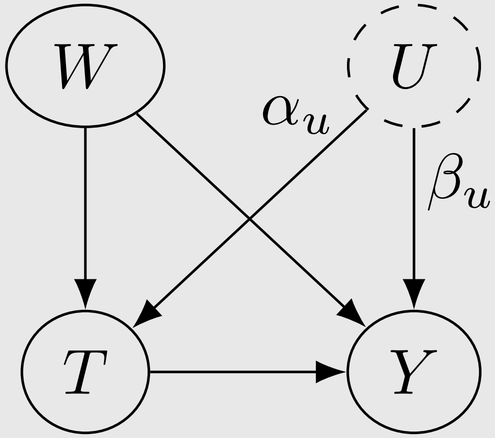
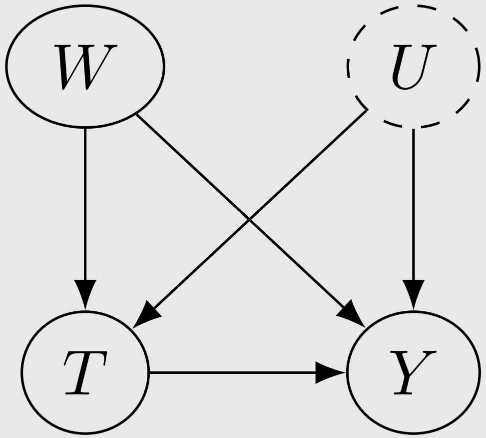


$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

Generalization to Arbitrary Linear SCMs

We've considered specifically the ATE in this simple graph

$$\mathbb{E}[Y(1) - Y(0)]$$



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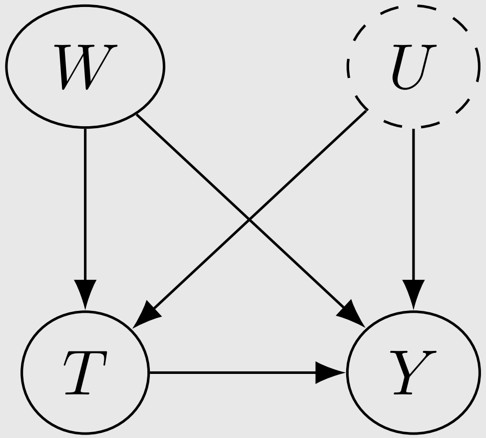

See "Sensitivity Analysis of Linear Structural Causal Models" from Cinelli et al. (2019) for arbitrary estimands in arbitrary graphs, where the structural equations are still linear

SCM:
$$T := \alpha_w W + \alpha_u U$$
$$Y := \beta_w W + \beta_u U + \delta T$$

Questions:

1. Given the above SCM, show that $\mathbb{E}_{W,U}\left[\mathbb{E}[Y \mid T=1, W, U] - \mathbb{E}[Y \mid T=0, W, U]\right] = \delta$
2. Does what we have shown in this section work if W is a vector?
3. How about if U is a vector?

Binary Treatment

Binary Treatment

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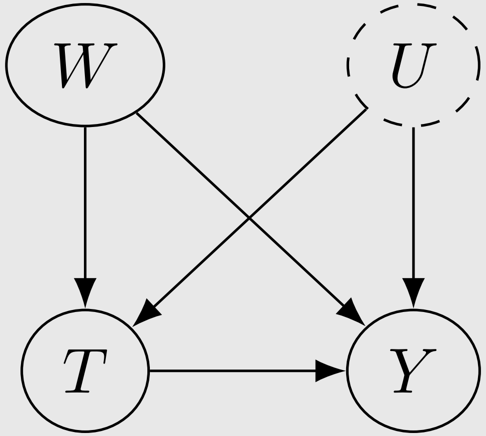

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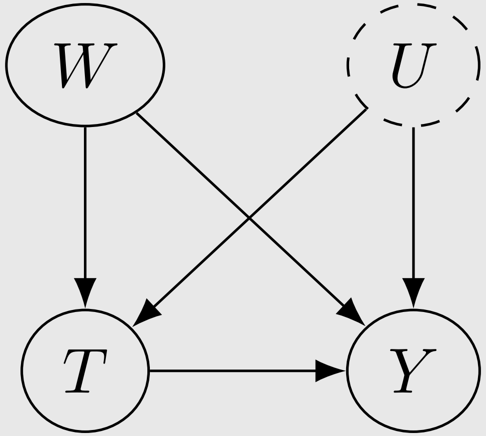

$$P(T = 1 \mid W, U) := \text{sigmoid}\left(\alpha_w W + \alpha_u U\right)$$
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Rosenbaum & Rubin (1983) and Imbens (2003)

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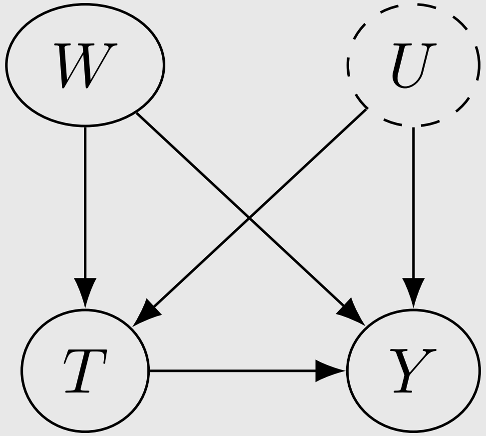

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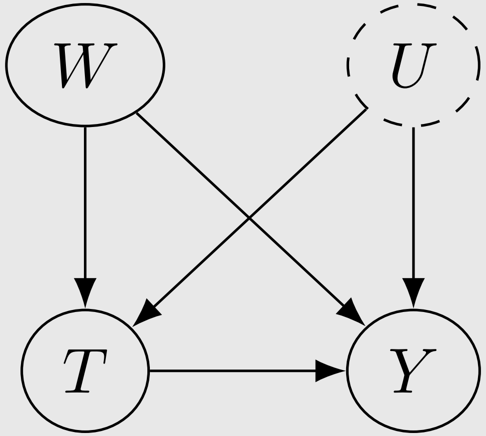

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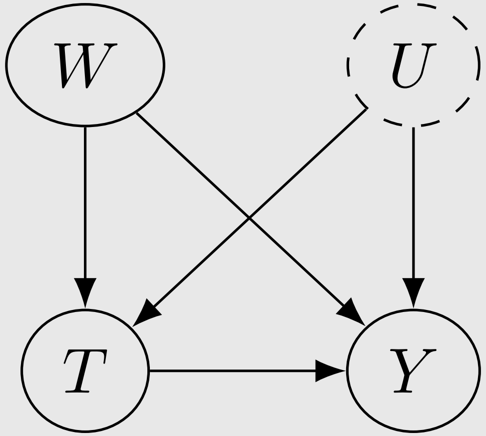

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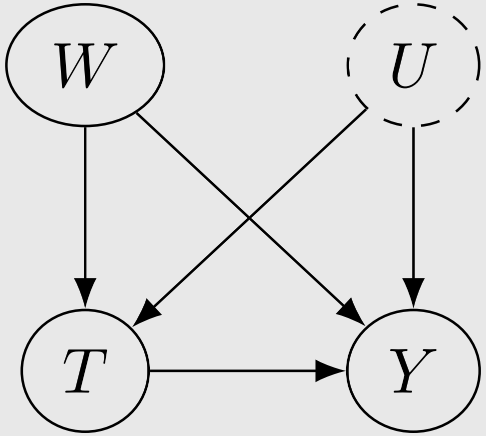

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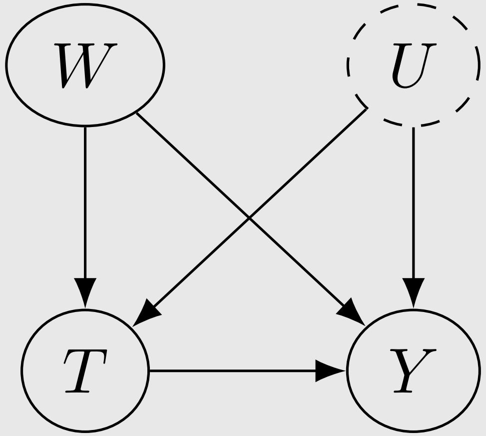

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Cinelli & Hazlett (2020) drop many of these assumptions

Making Sense of Sensitivity: Extending Omitted Variable Bias Cinelli & Hazlett (2020)

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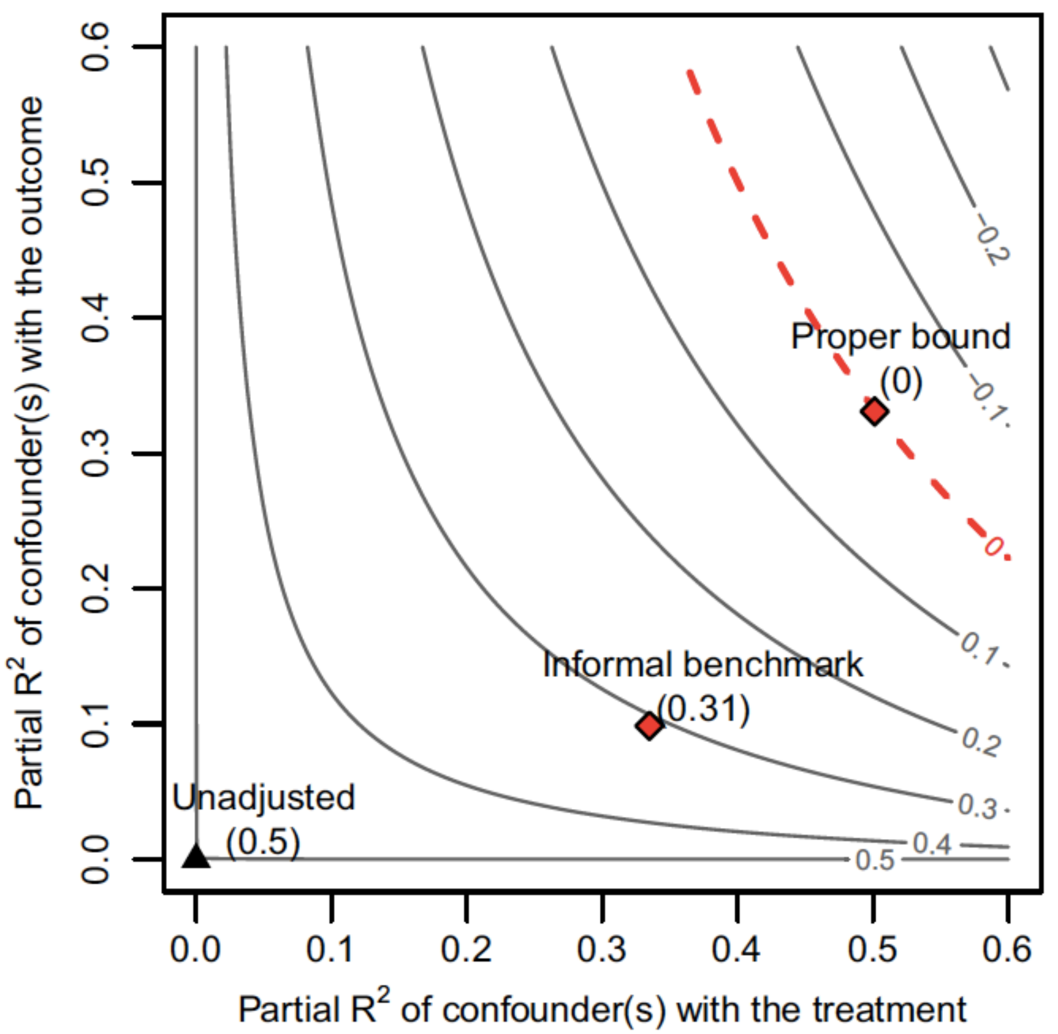

Figure 4:

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Imbens (2003) and follow-ups

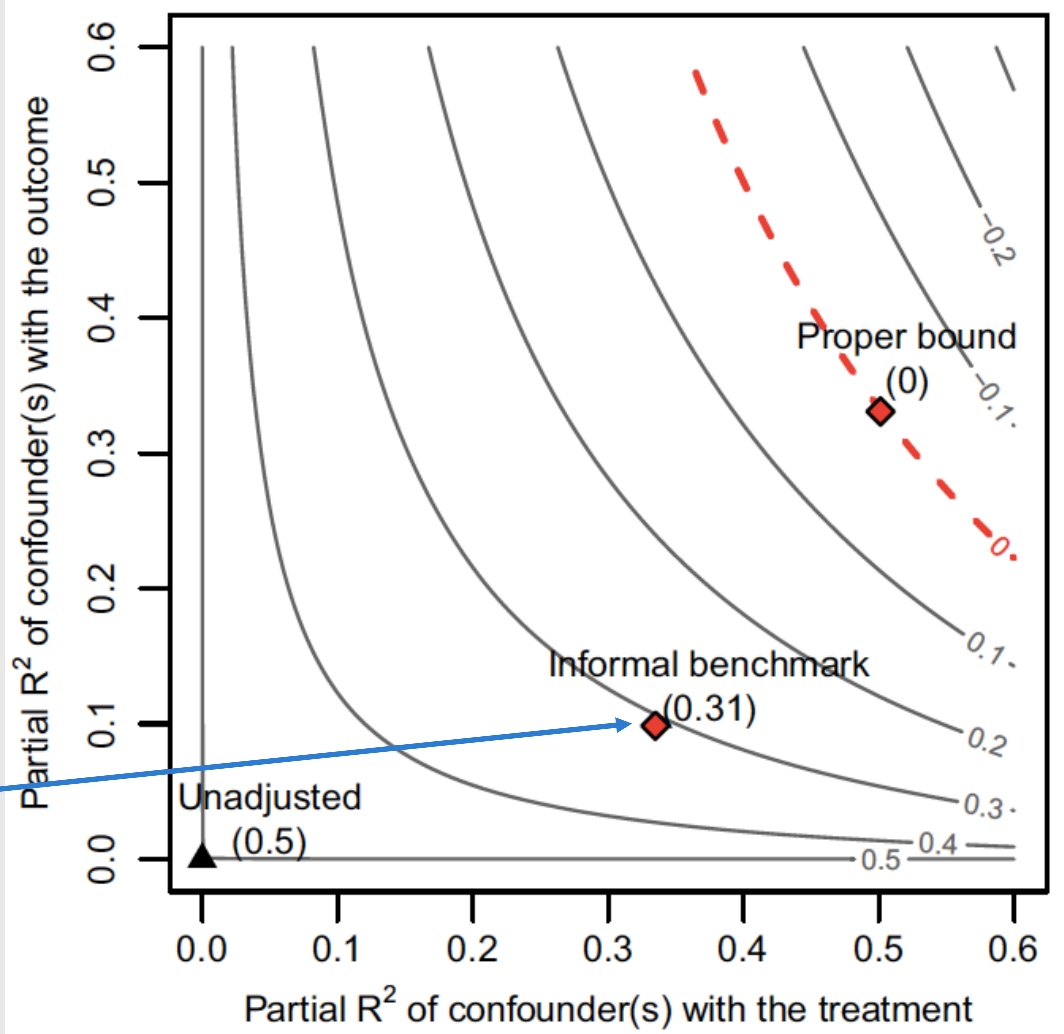

Figure 4:

Sense and Sensitivity Analysis Veitch & Zaveri (2020)

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Gives the same nice sensitivity curves we've been seeing, but with more flexible parameterization

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Both the treatment mechanism and the outcome mechanism can be modeled with **arbitrary machine learning models**, and we still get a closed-form expression for the bias (assuming well-specification)

Lots of Other Sensitivity Analysis Methods

- An Introduction to Sensitivity Analysis for Unobserved Confounding in Non-Experimental Prevention Research (Liu, Kuramoto, & Stuart., 2013)
- Rosenbaum has several (Rosenbaum 2002, 2010, 2017)
- Unmeasured Confounding for General Outcomes, Treatments, and Confounders (VanderWeele & Arah, 2011)
- Sensitivity Analysis Without Assumptions (Ding & VanderWeele, 2018)
- Flexible sensitivity analysis for observational studies without observable implications (Franks, D'Amour, & Feller, 2019)
- Bounds on the conditional and average treatment effect with unobserved confounding factors (Yadlowsky et al., 2018)